XXI. *A Memoir on the Single and Double Theta-Functions.*

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The Theta-Functions, although arising historically from the Elliptic Functions, may be considered as in order of simplicity preceding these, and connecting themselves directly with the exponential function (e^x or) exp. x; viz., they may be defined each of them as a sum of a series of exponentials, singly infinite in the case of the single functions, doubly infinite in the case of the double functions; and so on. The number of the single functions is $=4$; and the quotients of these, or say three of them each divided by the fourth, are the elliptic functions sn, cn, dn; the number of the double functions is ($4^2=$) 16; and the quotients of these, or say fifteen of them each divided by the sixteenth, are the hyper-elliptic functions of two arguments depending on the square root of a sextic function: generally the number of the p-tuple theta-functions is $=4^p$; and the quotients of these, or say all but one of them each divided by the remaining function, are the Abelian functions of p arguments depending on the irrational function y defined by the equation $\mathrm{F}(x, y)=0$ of a curve of deficiency p. If instead of connecting the ratios of the functions with a plane curve we consider the functions themselves as coordinates of a point in a (4^p-1)dimensional space, then we have the single functions as the four coordinates of a point on a quadri-quadric curve (one-fold locus) in ordinary space; and the double functions as the sixteen coordinates of a point on a quadri-quadric two-fold locus in 15-dimensional space, the deficiency of this two-fold locus being of course $=2$.

The investigations contained in the First Part of the present Memoir, although for simplicity of notation exhibited only in regard to the double functions are, in fact, applicable to the general case of the p-tuple functions; but in the main the Memoir relates only to the single and double functions, and the title has been given to it accordingly. The investigations just referred to extend to the single functions; and there is, it seems to me, an advantage in carrying on the two theories simultaneously up to and inclusive of the establishment of what I call the Product-theorem: this is a natural point of separation for the theories of the single and the double functions respectively. The ulterior developments of the two theories are indeed closely analogous to each other; but on the one hand the course of the single theory would be only with difficulty perceptible in the greater complexity of the double theory; and on the other hand we need the single theory as a guide for the course of the double theory.

I accordingly stop to point out in a general manner the course of the single theory, and, in connexion with it but more briefly, that of the double theory; and I then, in the Second and Third Parts respectively, consider in detail the two theories separately; first, that of the single functions, and then that of the double functions; the paragraphs of the Memoir are numbered consecutively.

The definition adopted for the theta-functions differs somewhat from that which is ordinarily used.

The earlier memoirs on the double theta-functions are the well-known ones :—

Rosenhain, "Mémoire sur les fonctions de deux variables et à quatre périodes, qui sont les inverses des intégrales ultra-elliptiques de la première classe." [1846.] Paris: 'Mém. Savans Étrang.' xi. (1851), pp. 361–468.

Göpel, 'Theoriæ transcendentium Abelianarum primi ordinis adumbratio levis. 'Crelle,' xxxv. (1847), pp. 277–312.

My first paper—Cayley, "On the Double θ-Functions in connexion with a 16-nodal Surface," 'Crelle-Borchardt,' lxxxiii. (1877), pp. 210–219—was founded directly upon these, and was immediately followed by Dr. Borchardt's paper,

Borchardt, "Ueber die Darstellung der *Kummer*sche Fläche vierter Ordnung mit sechzehn Knotenpunkte durch die *Göpel*schen Relation zwischen vier Theta-functionen mit zwei Variabeln." Ditto, pp. 220–233.

My other later papers are contained in the same Journal.

FIRST PART.—INTRODUCTORY.

Definition of the theta-functions.

1. The p-tuple functions depend upon $\frac{1}{2}p(p+1)$ parameters which are the coefficients of a quadric function of p ultimately disappearing integers, upon p arguments, and upon $2p$ characters, each $=0$ or 1, which form the characteristic of the 4^p functions; but it will be sufficient to write down the formulæ in the case $p=2$.

As already mentioned, the adopted definition differs somewhat from that which is ordinarily used. I use, as will be seen, a quadric function $\frac{1}{4}(a, h, b \between m, n)^2$ with *even* integer values of m, n, instead $(a, h, b \between m, n)^2$ with even or odd values; and I write the other term $\frac{1}{2}\pi i(mu+nv)$ instead of $mu+nv$; this comes to affecting the arguments u, v with a factor πi, so that the quarter periods (instead of being πi) are made to be $=1$.

2. We write

$$\begin{pmatrix} m, n \\ u, v \end{pmatrix} = \tfrac{1}{4}(a, h, b \between m, n)^2 + \tfrac{1}{2}\pi i(mu+nv),$$

and in like manner

$$\begin{pmatrix} m+\alpha, & n+\beta \\ u+\gamma, & v+\delta \end{pmatrix} = \tfrac{1}{4}(a,\ h,\ b \between m+\alpha,\ n+\beta)^2 + \tfrac{1}{2}\pi i\{(m+\alpha)(u+\gamma)(n+\beta)(v+\delta)\},$$

and prefixing to either of these the functional symbol exp. we have the exponential of the function in question, that is, e with the function as an exponent.

We then write, as the definition of the double theta-functions,

$$\vartheta\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}(u,\ v) = \Sigma \ \exp \begin{pmatrix} m+\alpha, & n+\beta \\ u+\gamma, & v+\delta \end{pmatrix},$$

where the summation extends to all positive and negative even integer values (zero included) of m and n respectively : α, β, γ, δ might denote any quantities whatever, but for the theta-functions they are regarded as denoting positive or negative integers ; this being so, it will appear that the only effect of altering each or any of them by an even integer is to reverse (it may be) the sign of the function ; and the distinct functions are consequently the $(4^2=)16$ functions obtained by giving to each of the quantities α, β, γ, δ the two values 0 and 1 successively.

3. We thus have the double theta-functions depending on the parameters $(a,\ h,\ b)$ which determine the quadric function $(a,\ h,\ b \between m,\ n)^2$ of the disappearing even integers $(m,\ n)$: and on the two arguments $(u,\ v)$: in the symbol $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$, which is called the characteristic, the characters α, β, γ, δ are each of them $=0$ or 1 ; and we thus have the 16 functions.

The parameters $(a,\ h,\ b)$ may be real or imaginary, but they must be such that reducing each of them to its real part the resulting function $(\ * \between m,\ n)^2$ is invariable in its sign, and negative for all real values of m and n : this is in fact the condition for the convergency of the series which give the values of the theta-functions.

4. The characteristic $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$ is said to be even or odd according as the sum $\alpha\gamma+\beta\delta$ is even or odd.

Allied functions.

5. As already remarked, the definition of

$$\vartheta\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}(u,\ v)$$

is not restricted to the case where the α, β, γ, δ represent integers, and there is actually occasion to consider functions of this form where they are not integers : in particular, α, β may be either or each of them of the form, integer $+\frac{1}{2}$. But the functions thus obtained *are not regarded as theta-functions,* and the expression theta-function will consequently not extend to include them.

Properties of the theta-functions : Various sub-headings.

Even-integer alteration of characters.

6. If x, y be integers, then m, n having the several even integer values from $-\infty$ to $+\infty$ respectively, it is obvious that $m+\alpha+2x$, $n+\beta+2y$ will have the same series of values with $m+\alpha$, $n+\beta$ respectively; and it thence follows that

$$\vartheta\begin{pmatrix}\alpha+2x, & \beta+2y \\ \gamma \quad , & \delta\end{pmatrix}(u,\ v)=\vartheta\begin{pmatrix}\alpha, & \beta \\ \gamma, & \delta\end{pmatrix}(u,\ v).$$

Similarly if z, w are integers, then in the function

$$\vartheta\begin{pmatrix}\alpha \quad , & \beta \\ \gamma+2z, & \delta+2w\end{pmatrix}(u,\ v)$$

the argument of the exponential function contains the term

$$\tfrac{1}{2}\pi i\{m+\alpha.u+\gamma+2z.+.n+\beta.v+\delta+2y\};$$

this differs from its original value by

$$\tfrac{1}{2}\pi i(m+\alpha.2z.+.n+\beta.2w),$$
$$=\pi i(mz+nw)+\pi i(\alpha z+\beta w),$$

and then, m and n being even integers, $mz+nw$ is also an even integer, and the term $\pi i(mz+nw)$ does not affect the value of the exponential: we thus introduce into each term of the series the factor exp. $\pi i(\alpha z+\beta w)$, which is in fact $=(-)^{\alpha z+\beta w}$; and we consequently have

$$\vartheta\begin{pmatrix}\alpha \quad , & \beta \\ \gamma+2z, & \delta+2w\end{pmatrix}(u,\ v)=(-)^{\alpha z+\beta w}\vartheta\begin{pmatrix}\alpha, & \beta \\ \gamma, & \delta\end{pmatrix}(u,\ v);$$

or, uniting the two results,

$$\vartheta\begin{pmatrix}\alpha+2x, & \beta+2y \\ \gamma+2z, & \delta+2w\end{pmatrix}(u,\ v)=(-)^{\alpha z+\beta w}\vartheta\begin{pmatrix}\alpha, & \beta \\ \gamma, & \delta\end{pmatrix}(u,\ v):$$

this sustains the before-mentioned conclusion that the only distinct functions are the 16 functions obtained by giving to the characters α, β, γ, δ the values 0 and 1 respectively.

Odd-integer alteration of characters.

7. The effect is obviously to interchange the different functions.

Even and odd functions.

8. It is clear that $-m-\alpha$, $-n-\beta$ have precisely the same series of values with $m+\alpha$, $n+\beta$ respectively: hence considering the function

$$\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(-u,\ -v)$$

the linear term in the argument of the exponential may be taken to be

$$\tfrac{1}{2}\pi i\{-m-\alpha.-u+\gamma.+.-n-\beta.-v+\delta\},$$

which is

$$=\tfrac{1}{2}\pi i\{m+\alpha.u+\gamma.+.n+\beta.v+\delta\}-\pi i\{m+\alpha.\gamma.+.n+\beta.\delta\};$$

the second term is here $=-\pi i(m\gamma+n\delta)-\pi i(\alpha\gamma+\beta\delta)$, where $m\gamma+n\delta$ being an even integer the part $-\pi i(m\gamma+n\delta)$ does not alter the value of the exponential: the effect of the remaining part $-\pi i(\alpha\gamma+\beta\delta)$ is to affect each term of the series with the factor exp. $-\pi i(\alpha\gamma+\beta\delta)$, or what is the same thing, exp. $\pi i(\alpha\gamma+\beta\delta)$, each of these being in fact $=(-)^{\alpha\gamma+\beta\delta}$.

We have thus

$$\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(-u,-v)=(-)^{\alpha\gamma+\beta\delta}\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u,\ v),$$

viz., $\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u,\ v)$ is an even or odd function of the two arguments $(u,\ v)$ conjointly, according as the characteristic $\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}$ is even or odd.

The quarter-periods unity.

9. Taking z and w integers, we have from the definition

$$\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u+z,\ v+w)=\vartheta\begin{pmatrix}\alpha & , \beta\\ \gamma+z, & \delta+w\end{pmatrix}(u,\ v),$$

viz., the effect of altering the arguments u, v into $u+z$, $v+w$ is simply to interchange the functions as shown by this formula.

If z and w are each of them even, then replacing them by $2z$, $2w$ respectively, we have

$$\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u+2z,\ v+2w)=\vartheta\begin{pmatrix}\alpha & , \beta\\ \gamma+2z, & \delta+2w\end{pmatrix}(u,\ v),$$

which by a preceding formula is

$$=(-)^{\alpha z+\beta w}\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u,\ v),$$

or the function is altered at most in its sign. And again writing $2z$, $2w$ for z, w we have

$$\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u+4z,\ v+4w)=\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u,\ v).$$

In reference to the foregoing results we say that the theta-functions have the quarter-periods (1, 1), the half-periods (2, 2), and the whole periods (4, 4).

The conjoint quarter quasi-periods.

10. Taking x, y integers, we consider the effect of the change, u, v into

$$u+\frac{1}{\pi i}(ax+hy),\ v+\frac{1}{\pi i}(hx+by).$$

It is convenient to start from the function

$$\vartheta\begin{pmatrix}\alpha-x, & \beta-y\\ \gamma & ,\ \delta\end{pmatrix}\left(u+\frac{1}{\pi i}(ax+hy),\ v+\frac{1}{\pi i}(hx+by)\right);$$

the argument of the exponential is here

$$\tfrac{1}{4}(a,\ h,\ b)(m+\alpha-x,\ n+\beta-y)^2$$
$$+\tfrac{1}{2}\pi i\left\{m+\alpha-x.u+\gamma+\frac{1}{\pi i}(ax+hy).+.n+\beta-y.v+\delta+\frac{1}{\pi i}(hx+by)\right\}$$

which is

$$=\tfrac{1}{4}(a,\ h,\ b)(m+\alpha,\ n+\beta)^2+\tfrac{1}{2}\pi i(m+\alpha.u+\gamma.+.n+\beta.v+\delta)$$

$+$ other terms which are as follows: viz., they are

$$\begin{array}{ll}-\tfrac{1}{2}(a,\ h,\ b)(m+\alpha,\ n+\beta)(x,\ y) & +\tfrac{1}{2}(m+\alpha.ax+hy.+.n+\beta.hx+by)\\ +\tfrac{1}{4}(a,\ h,\ b)(x,\ y)^2 & -\tfrac{1}{2}\pi i(x.u+\gamma.+.y.v+\delta)\\ & -\tfrac{1}{2}(x.ax+hy.+.y.hx+by),\end{array}$$

where the terms of the right hand column are in fact

$$=+\tfrac{1}{2}(a,\ h,\ b)(m+\alpha,\ n+\beta)(x,\ y)$$
$$-\tfrac{1}{2}\pi i(x.u+\gamma.+.y.v+\delta)$$
$$-\tfrac{1}{2}(a,\ h,\ b)(x,\ y)^2,$$

and the other terms in question thus reduce themselves to

$$-\tfrac{1}{4}(a,\ h,\ b \between x,\ y)^2 \qquad -\tfrac{1}{2}\pi i(x.u+\gamma.+.y.v+\delta),$$

which are independent of m, n, and they thus affect each term of the series with the same exponential factor. The result is

$$\begin{aligned}&\vartheta\begin{pmatrix}\alpha-x, & \beta-y\\ \gamma & ,\ \delta\end{pmatrix}\left(u+\frac{1}{\pi i}(ax+hy),\ v+\frac{1}{\pi i}(hx+by)\right)\\ &=\exp\left\{-\tfrac{1}{4}(a,\ h,\ b \between x,\ y)^2-\tfrac{1}{2}\pi i(x.u+\gamma.+.y.v+\delta)\right\}.\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u,\ v);\end{aligned}$$

or (what is the same thing) for α, β, writing $\alpha+x$, $\beta+y$ respectively, we have

$$\begin{aligned}&\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma & \delta\end{pmatrix}\left(u+\frac{1}{\pi i}(ax+hy),\ v+\frac{1}{\pi i}(hx+by)\right)\\ &=\exp\left\{-\tfrac{1}{4}(a,\ h,\ b \between x,\ y)^2-\tfrac{1}{2}\pi i(x.u+\gamma.+.y.v+\delta)\right\}.\vartheta\begin{pmatrix}\alpha+x, & \beta+y\\ \gamma & ,\ \delta\end{pmatrix}(u,\ v).\end{aligned}$$

Taking x, y even, or writing $2x$, $2y$ for x, y, then on the right hand side we have

$$\vartheta\begin{pmatrix}\alpha+2x, & \beta+2y\\ \gamma & ,\ \delta\end{pmatrix}(u,\ v),\ \text{which is}=\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u,\ v),$$

but there is still the exponential factor.

11. The formulæ show that the effect of the change u, v into $u+\frac{1}{\pi i}(ax+hy)$, $v+\frac{1}{\pi i}(hx+by)$, where x, y are integers, is to interchange the functions, affecting them however with an exponential factor; and we hence say that $\frac{1}{\pi i}(a,\ h)$, $\frac{1}{\pi i}(h,\ b)$ are conjoint quarter quasi-periods.

The product-theorem.

12. We multiply two theta-functions

$$\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u+u',\ v+v'),\ \vartheta\begin{pmatrix}\alpha', & \beta'\\ \gamma', & \delta'\end{pmatrix}(u-u',\ v-v');$$

it is found that the result is a sum of four products

$$\Theta\begin{pmatrix}\frac{1}{2}(\alpha+\alpha')+p, & \frac{1}{2}(\beta+\beta')+q\\ \gamma+\gamma' & ,\ \delta+\delta'\end{pmatrix}(2u,\ 2v)\quad.\quad\Theta\begin{pmatrix}\frac{1}{2}(\alpha-\alpha')+p, & \frac{1}{2}(\beta-\beta')+q\\ \gamma-\gamma' & ,\ \delta-\delta'\end{pmatrix}(2u',\ 2v'),$$

where p, q have in the four products respectively the values (0, 0), (1, 0), (0, 1), and (1, 1); Θ is written in place of ϑ to denote that the parameters (a, h, b) are to be changed into $(2a, 2h, 2b)$. It is to be noticed that if α, α' are both even or both odd then $\tfrac{1}{2}(\alpha+\alpha')$, $\tfrac{1}{2}(\alpha-\alpha')$ are integers; and so if β, β' are both even or both odd then $\tfrac{1}{2}(\beta+\beta')$, $\tfrac{1}{2}(\beta-\beta')$ are integers; and these conditions being satisfied (and in particular they are so if $\alpha=\alpha'$, $\beta=\beta'$) then the functions on the right hand side of the equation are theta-functions (with new parameters as already mentioned); but if the conditions are not satisfied, then the functions on the right hand side are only allied functions. In the applications of the theorem the functions on the right hand side are eliminated between the different equations, as will appear.

13. The proof is immediate: in the first of the theta-functions the argument of the exponential is

$$\begin{pmatrix} m+\alpha & , n+\beta \\ u+u'+\gamma, & v+v'+\delta \end{pmatrix},$$

and in the second, writing m', n' instead of m, n, the argument is

$$\begin{pmatrix} m'+\alpha' & , n'+\beta' \\ u-u'+\gamma', & v-v'+\delta' \end{pmatrix},$$

hence in the product, the argument of the exponential is the sum of these two functions,

$$\begin{aligned} &=\tfrac{1}{4}(a, h, b \between m+\alpha,\ n+\beta)^2+\tfrac{1}{2}\pi i(m+\alpha\,.\,u+u'+\gamma\,.+.\,n+\beta\,.\,v+v'+\delta) \\ &+\tfrac{1}{4}(a, h, b \between m'+\alpha', n'+\beta')^2+\tfrac{1}{2}\pi i(m'+\alpha'.\,u-u'+\gamma'.+.\,n'+\beta'.\,v-v'+\delta'). \end{aligned}$$

Comparing herewith the sum of the two functions

$$\begin{pmatrix} \mu+\tfrac{1}{2}(\alpha+\alpha'), & \nu+\tfrac{1}{2}(\beta+\beta') \\ 2u+\gamma+\gamma' & , 2v+\delta+\delta' \end{pmatrix}, \quad \begin{pmatrix} \mu'+\tfrac{1}{2}(\alpha-\alpha'), & \nu'+\tfrac{1}{2}(\beta-\beta') \\ 2u'+\gamma-\gamma' & , 2\nu'+\delta-\delta' \end{pmatrix},$$

$$\begin{aligned} &=\tfrac{1}{4}(2a, 2h, 2b \between \mu+\tfrac{1}{2}(\alpha+\alpha'), \nu+\tfrac{1}{2}(\beta+\beta'))^2 \\ &\qquad\qquad +\tfrac{1}{2}\pi i\{\mu+\tfrac{1}{2}(\alpha+\alpha').2u+\gamma+\gamma'.+.\nu+\tfrac{1}{2}(\beta+\beta').2v+\delta+\delta'\} \\ &+\tfrac{1}{4}(2a, 2h, 2b \between \mu'+\tfrac{1}{2}(\alpha-\alpha'), \nu'+\tfrac{1}{2}(\beta-\beta'))^2 \\ &\qquad\qquad +\tfrac{1}{2}\pi i\{\mu'+\tfrac{1}{2}(\alpha-\alpha').2u'+\gamma-\gamma'.+.\nu'+\tfrac{1}{2}(\beta-\beta').2v'+\delta-\delta'\}, \end{aligned}$$

the two sums are identical if only

$$\begin{aligned} m+m'&=2\mu,\ n+n'=2\nu, \\ m-m'&=2\mu',\ n-n'=2\nu', \end{aligned}$$

as may easily be verified by comparing the quadric and linear terms separately. The product of the two theta-functions is thus

$$=\Sigma \exp \begin{pmatrix} \mu+\tfrac{1}{2}(\alpha+\alpha'), & \nu+\tfrac{1}{2}(\beta+\beta') \\ 2u+\gamma+\gamma' & , 2v+\delta+\delta' \end{pmatrix} \;.\; \Sigma \exp \begin{pmatrix} \mu'+\tfrac{1}{2}(\alpha-\alpha'), & \nu'+\tfrac{1}{2}(\beta-\beta') \\ 2u'+\gamma-\gamma' & , 2v'+\delta-\delta' \end{pmatrix},$$

with the proper conditions as to the values of μ, ν and of μ', ν' in the two sums respectively. As to this, observe that m, m' are even integers; say for a moment that they are similar when they are both $\equiv 0$ or both $\equiv 2$ (mod 4), but dissimilar when they are one of them $\equiv 0$ and the other of them $\equiv 2$ (mod 4); and the like as regards n, n'. Hence if m, m' are similar μ, μ' are both of them even; but if m, m' are dissimilar then μ, μ' are both of them odd. And so if n, n' are similar, ν, ν' are both of them even, but if n, n' are dissimilar then ν, ν' are both odd.

14. There are four cases

m, m' similar, n, n' similar,
m, m' dissimilar, n, n' similar,
m, m' similar, n, n' dissimilar,
m, m' dissimilar, n, n' dissimilar,

and in the first of these μ, ν, μ', ν' are all of them even, and the product is

$$=\Theta\begin{pmatrix} \tfrac{1}{2}(\alpha+\alpha'), & \tfrac{1}{2}(\beta+\beta') \\ \gamma+\gamma', & \delta+\delta' \end{pmatrix}(2u,\ 2v) \;.\; \Theta\begin{pmatrix} \tfrac{1}{2}(\alpha-\alpha'), & \tfrac{1}{2}(\beta-\beta') \\ \gamma-\gamma', & \delta-\delta' \end{pmatrix}(2u',\ 2v').$$

In the second case, writing $\mu+1$, $\mu'+1$ for μ, μ' the new values of μ, μ' will be both even, and we have the like expression with only the characters $\tfrac{1}{2}(\alpha+\alpha')$, $\tfrac{1}{2}(\alpha-\alpha')$ each increased by 1; so in the third case we obtain the like expression with only the characters $\tfrac{1}{2}(\beta+\beta')$, $\tfrac{1}{2}(\beta-\beta')$ each increased by 1; and in the fourth case the like expression with the four upper characters each increased by 1. The product of the two theta-functions is thus equal to the sum of the four products, according to the theorem.

Résumé of the ulterior theory of the single functions.

15. For the single theta-functions the Product-theorem comprises 16 equations, and for the double theta-functions, 256 equations: these systems will be given in full in the sequel. But attending at present to the single functions, I write down here the first four of the 16 equations, viz.: these are

$$\begin{array}{lllllr}
0.0 & \vartheta\begin{pmatrix}0\\0\end{pmatrix}(u+u').\vartheta\begin{pmatrix}0\\0\end{pmatrix}(u-u') & & & = & XX'+YY', \\
1.0 & \vartheta\,{}^{1}_{0} & \text{,,} & \vartheta\,{}^{1}_{0} \quad \text{,,} & = & YX'+XY', \\
0.1 & \vartheta\,{}^{0}_{1} & \text{,,} & \vartheta\,{}^{0}_{1} \quad \text{,,} & = & XX'-YY', \\
1.1 & \vartheta\,{}^{1}_{1} & \text{,,} & \vartheta\,{}^{1}_{1} \quad \text{,,} & = & -YX'+XY';
\end{array}$$

where X, Y denote $\Theta\begin{pmatrix}0\\0\end{pmatrix}(2u)$, $\Theta\begin{pmatrix}1\\0\end{pmatrix}(2u)$ respectively, and X′, Y′ the same functions of $2u'$ respectively. In the other equations we have on the left hand the product of *different* theta-functions of $u+u'$, $u-u'$ respectively, and on the right hand expressions involving other functions, X_1, Y_1, X_1', Y_1', &c., of $2u$ and $2u'$ respectively.

16. By writing $u'=0$, we have on the left hand, squares or products of theta-functions of u, and on the right hand expressions containing functions of $2u$: in particular the above equations show that the squares of the four theta-functions are equal to linear functions of X, Y; that is, there exist between the squared functions two linear relations: or again, introducing a variable argument x, the four squared functions may be taken to be proportional to linear functions

$$\mathfrak{A}(a-x),\ \mathfrak{B}(b-x),\ \mathfrak{C}(c-x),\ \mathfrak{D}(d-x)$$

where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, a, b, c, d, are constants. This suggests a new notation for the four functions, viz.: we write

$$\begin{array}{rcccc}
& \vartheta\begin{pmatrix}0\\0\end{pmatrix}(u), & \vartheta\begin{pmatrix}1\\0\end{pmatrix}(u), & \vartheta\begin{pmatrix}0\\1\end{pmatrix}(u), & \vartheta\begin{pmatrix}1\\1\end{pmatrix}(u) \\
= & Au, & Bu, & Cu, & Du\,;
\end{array}$$

and the result just mentioned then is

$$\begin{array}{rcccccc}
& A^2u & : & B^2u & : & C^2u & : \quad D^2u \\
= & \mathfrak{A}(a-x) & : & \mathfrak{B}(b-x) & : & \mathfrak{C}(c-x) & : \mathfrak{D}(d-x),
\end{array}$$

which expresses that the four functions are the coordinates of a point on a quadri-quadric curve in ordinary space.

17. The remaining 12 of the 16 equations then contain on the left hand products such as $A(u+u').B(u-u')$; and by suitably combining them we obtain equations such as

$$\frac{\overset{u+u'}{\mathrm{B}}\,.\,\overset{u-u'}{\mathrm{A}}\;-\;\overset{u+u'}{\mathrm{A}}\,.\,\overset{u-u'}{\mathrm{B}}}{\mathrm{C}\,.\,\mathrm{D}\;+\;\mathrm{D}\,.\,\mathrm{C}}=\text{function }(u'),$$

where for brevity the arguments are written above; viz., the numerator of the fraction is

$$\mathrm{B}(u+u')\mathrm{A}(u-u')-\mathrm{A}(u+u')\mathrm{B}(u-u'),$$

and its denominator is

$$\mathrm{C}(u+u')\mathrm{D}(u-u')+\mathrm{D}(u+u')\mathrm{C}(u-u').$$

Admitting the form of the equation, the value of the function of u' is at once found by writing in the equation $u=0$; it is, as it ought to be, a function vanishing for $u'=0$.

18. Take in this equation u' indefinitely small; each side divides by u', and the resulting equation is

$$\frac{\mathrm{A}u\mathrm{B}'u-\mathrm{B}u\mathrm{A}'u}{\mathrm{C}u\mathrm{D}u}=\text{const.}$$

where $\mathrm{A}'u$, $\mathrm{B}'u$ are the derived functions, or differential coefficients in regard to u. It thus appears that the combination $\mathrm{A}u\mathrm{B}'u-\mathrm{B}u\mathrm{A}'u$ is a constant multiple of $\mathrm{C}u\mathrm{D}u$: or, what is the same thing, that the differential coefficient of the quotient-function $\frac{\mathrm{B}u}{\mathrm{A}u}$ is a constant multiple of the product of the two quotient-functions $\frac{\mathrm{C}u}{\mathrm{A}u}$ and $\frac{\mathrm{D}u}{\mathrm{A}u}$.

19. And then substituting for the several quotient-functions their values in terms of x, we obtain a differential relation between x, u; viz.: the form hereof is

$$du=\frac{\mathrm{M}dx}{\sqrt{a-x.b-x.c-x.d-x}},$$

and it thus appears that the quotient-functions are in fact elliptic-functions: the actual values as obtained in the sequel are

$$\begin{aligned}
\operatorname{sn}\mathrm{K}u &= -\frac{1}{\sqrt{k}}\mathrm{D}u\div\mathrm{C}u,\\
\operatorname{cn}\mathrm{K}u &= \sqrt{\frac{k'}{k}}\mathrm{B}u\div\mathrm{C}u,\\
\operatorname{dn}\mathrm{K}u &= \sqrt{k'}\mathrm{A}u\div\mathrm{C}u\,;
\end{aligned}$$

and we thus of course identify the functions $\mathrm{A}u$, $\mathrm{B}u$, $\mathrm{C}u$, $\mathrm{D}u$ with the H and Θ of Jacobi.

20. If in the above-mentioned four equations we write first $u=0$, and then $u'=0$, and by means of the results eliminate from the original equations the quantities

X, Y, X′, Y′ which occur therein, we obtain expressions for the four products such as $A(u+u')A(u-u')$. One of these equations is

$$C^2 0 . C(u+u')C(u-u') = C^2 u C^2 u' - D^2 u D^2 u'.$$

Taking herein u' indefinitely small, we obtain

$$\frac{Cu C''u - (C'u)^2}{C^2 u} = \frac{C''0}{C0} - \left(\frac{D'0}{C0}\right)^2 \frac{D^2 u}{C^2 u},$$

where the left hand side is in fact $\frac{d^2}{du^2}\log Cu$, or this second derived function of the theta-function Cu is given in terms of the quotient-function $\frac{Du}{Cu}$: hence integrating twice, and taking the exponential of each side we obtain Cu as an exponential the argument of which contains the double integral $\iint \frac{D^2 u}{C^2 u}(du)^2$, of a squared quotient-function. This in fact corresponds to JACOBI's equation

$$\Theta u = \sqrt{\frac{2Kk'}{\pi}}\, e^{\frac{1}{2}u^2\left(1-\frac{E}{K}\right) - k^2 \int_0 du \int_0 du \operatorname{sn}^2 u}.$$

21. From the same equation $C^2 0 . C(u+u')C(u-u') = C^2 u C^2 u' - D^2 u D^2 u'$, differentiating logarithmically in regard to u', and integrating in regard to u, we obtain an equation containing on the left hand side a term $\log \frac{C(u-u')}{C(u+u')}$, and on the right hand an integral in regard to u, and which in fact corresponds to JACOBI's equation

$$u\frac{\Theta' a}{\Theta a} + \tfrac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)} = \Pi(u, a),$$

$$= \int_0 \frac{k^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \operatorname{sn}^2 u\, du}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u}.$$

22. It may further be noticed that if in the equation in question, and in the three other equations of the system, we introduce into the integral the variable x in place of u, and the corresponding quantity ξ in place of u', then the integral is that of an expression such as

$$\frac{dx}{T\sqrt{a-x.b-x.c-x.d-x}},$$

where T is $= x-\xi$, or is $=$ any one of three forms such as

$$\begin{vmatrix} 1, & x+\xi, & x\xi \\ 1, & a+b, & ab \\ 1, & c+d, & cd \end{vmatrix}$$

Résumé of the ulterior theory of the double functions.

23. The ulterior theory of the double functions is intended to be carried out on the like plan. As regards these, it is to be observed here that we have not only the 16 equations leading to linear relations between the squared functions, but that the remaining 240 equations lead also to linear relations between binary products of different functions. We have thus between the 16 functions a system of quadric relations, which in fact determine the ratios of the 16 functions in terms of two variable parameters x, y. (The 16 functions are thus the coordinates of a point on a quadri-quadric two-fold locus in 15-dimensional space.) The forms depend upon six constants, a, b, c, d, e, f: and writing for shortness

$$\sqrt{a} = \qquad \sqrt{a-x.a-y},$$

$$\vdots$$

$$\sqrt{ab} = \frac{1}{x-y}\{\sqrt{a-x.b-x.f-x.c-y.d-y.e-y} + \sqrt{a-y.b-y.f-y.c-x.d-x.e-x}\},$$

$$\vdots$$

(observe that in the symbols $\sqrt{ab}$ it is always f that accompanies the two expressed letters a, b—or, what is the same thing, the duad ab is really an abbreviation for the double triad $abf.cde$); then the 16 functions are proportional to properly determined constant multiples of

$$\sqrt{a},\ \sqrt{b},\ \sqrt{c},\ \sqrt{d},\ \sqrt{e},\ \sqrt{f},\ \sqrt{ab},\ \sqrt{ac},\ \sqrt{ad},\ \sqrt{ae},\ \sqrt{bc},\ \sqrt{bd},\ \sqrt{be},\ \sqrt{cd},\ \sqrt{ce},\ \sqrt{de},$$

and this suggests that the functions shall be represented by the single and double letter notation $\mathrm{A}(u, v)$, . . $\mathrm{AB}(u, v)$ . . . viz., if for shortness the arguments are omitted, then we have

$$\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \mathrm{D},\ \mathrm{E},\ \mathrm{F},\ \mathrm{AB},\ \mathrm{AC},\ \mathrm{AD},\ \mathrm{AE},\ \mathrm{BC},\ \mathrm{BD},\ \mathrm{BE},\ \mathrm{CD},\ \mathrm{CE},\ \mathrm{DE}$$

proportional to determinate constant multiples of the before-mentioned functions $\sqrt{a}$, . . $\sqrt{ab}$, . . . of x and y.

24. It is interesting to notice why in the expressions for $\sqrt{ab}$, &c., the sign connecting the two radicals is $+$; the effect of the interchange of x, y is in fact to change (u, v) into $(-u, -v)$; consequently to change the sign of the odd functions, and to leave unaltered those of the even functions: the interchange does in fact leave $\sqrt{a}$, &c., unaltered, while it changes $\sqrt{ab}$, &c., into $-\sqrt{ab}$, &c.; and thus, since only the ratios are attended to, there is a change of sign as there should be.

25. The equations of the product-theorem lead to expressions for

$$\overset{u+u'}{\mathrm{A}} . \overset{u-u'}{\mathrm{B}} - \overset{u+u'}{\mathrm{B}} . \overset{u-u'}{\mathrm{A}},$$

(where the arguments, written above, are used to denote the *two* arguments, viz.: $u+u'$ to denote $(u+u',\ v+v')$ and $u-u'$ to denote $(u-u',\ v-v')$; and where the letters A, B denote each or either of them a single or double letter) in terms of the functions of (u, v) and of (u', v'): and in any such expression taking u', v' each of them indefinitely small, but with their ratio arbitrary, we obtain the value of

$$\overset{u}{\mathrm{A}}.\eth\overset{u}{\mathrm{B}}-\overset{u}{\mathrm{B}}.\eth\overset{u}{\mathrm{A}},$$

(viz., u here stands for the two arguments (u, v), and $\eth$ denotes total differentiation $\eth\mathrm{A}=du\frac{d}{du}\mathrm{A}(u,\ v)+dv\frac{d}{dv}\mathrm{A}(u,\ v)$) as a quadric function of the functions of (u, v): or dividing by A^2, the form is $\eth\frac{\mathrm{B}}{\mathrm{A}}=$ a function of the quotient-functions $\frac{\mathrm{B}}{\mathrm{A}}$, &c., that is, we have the differentials of the quotient-functions in terms of the quotient-functions themselves. Substituting for the quotient-functions their values in terms of x, y, we should obtain the differential relations between dx, dy, du, dv, viz., putting for shortness $\mathrm{X}=a-x.b-x.c-x.d-x.e-x.f-x$, and $\mathrm{Y}=a-y.b-y.c-y.d-y.e-y.f-y$, these are of the form

$$\frac{dx}{\sqrt{\mathrm{X}}}-\frac{dy}{\sqrt{\mathrm{Y}}},\ \frac{xdx}{\sqrt{\mathrm{X}}}-\frac{ydy}{\sqrt{\mathrm{Y}}},$$

each of them equal to a linear function of du and dv: so that the quotient-functions are in fact the 15 hyper-elliptic functions belonging to the integrals $\int\frac{dx}{\sqrt{\mathrm{X}}}$, $\int\frac{xdx}{\sqrt{\mathrm{X}}}$; and there is thus an addition-theorem for them, in accordance with the theory of these integrals.

26. The first 16 equations of the product-theorem, putting therein first $u=0$, $v=0$, and then $u'=0$, $v'=0$, and using the results to eliminate the functions on the right hand side, give expressions for

$$\overset{u+u'}{\mathrm{A}}\ .\ \overset{u-u'}{\mathrm{B}},\ \&\text{c.}$$

(that is, $\mathrm{A}(u+u',\ v+v').\mathrm{B}(u-u',\ v-v')$ &c.) in terms of the functions of (u, v) and (u', v'): and we have thus an addition-with-subtraction theorem for the double theta-functions. And we have thence also consequences analogous to those which present themselves in the theory of the single functions.

Remark as to notation.

27. I remark as regards the single theta-functions that the characteristics

$$\begin{pmatrix}0\\0\end{pmatrix}\quad\begin{pmatrix}1\\0\end{pmatrix}\quad\begin{pmatrix}0\\1\end{pmatrix}\quad\begin{pmatrix}1\\1\end{pmatrix}$$

might for shortness be represented by a series of current numbers

$$0,\qquad 1,\qquad 2,\qquad 3$$

and the functions be accordingly called $\vartheta_0 u$, $\vartheta_1 u$, $\vartheta_2 u$, $\vartheta_3 u$; but that instead of this I prefer to use throughout the before-mentioned functional symbols

$$\mathrm{A},\qquad \mathrm{B},\qquad \mathrm{C},\qquad \mathrm{D}.$$

As regards the double functions, I do, however, denote the characteristics

$$\begin{array}{cccc|cccc|cccc|cccc|}
00, & 10, & 01, & 11 & 00, & 10, & 01, & 11 & 00, & 10, & 01, & 11 & 00, & 10, & 01, & 11 \\
00 & 00 & 00 & 00 & 10 & 10 & 10 & 10 & 01 & 01 & 01 & 01 & 11 & 11 & 11 & 11
\end{array}$$

by a series of current numbers

$$0,\quad 1,\quad 2,\quad 3,\qquad 4,\quad 5,\quad 6,\quad 7,\qquad 8,\quad 9,\quad 10,\quad 11,\qquad 12,\quad 13,\quad 14,\quad 15$$

and write the functions as ϑ, $\vartheta_1 \ldots \vartheta_{15}$ accordingly; and I use also, as and when it is convenient, the foregoing single and double letter notation A, AB, . . , which correspond to them in the order

BD, CE, CD, BE, AC, C, AB, B, BC, DE, F, A, AD, D, E, AE

Moreover I write down for the most part a single argument only: thus, $\mathrm{A}(u+u')$ stands for $\mathrm{A}(u+u', v+v')$, $\mathrm{A}(0)$ for $\mathrm{A}(0, 0)$: and so in other cases.

SECOND PART.—THE SINGLE THETA-FUNCTIONS.

Notation, &c.

28. Writing exp. $a=q$, and converting the exponentials into circular functions, we have directly from the definition

$$\vartheta{}^{0}_{0}(u)=\vartheta u=\mathrm{A}u=1+2q\cos\pi u+2q^4\cos 2\pi u+2q^9\cos 3\pi u+\ldots\;,$$

$$\vartheta{}^{1}_{0}(u)=\vartheta_1 u=\mathrm{B}u=\quad 2q^{\frac{1}{4}}\cos\tfrac{1}{2}\pi u+2q^{\frac{9}{4}}\cos\tfrac{3}{2}\pi u+2q^{\frac{25}{4}}\cos\tfrac{5}{2}\pi u+\ldots,$$

$$\vartheta{}^{0}_{1}(u)=\vartheta_2 u=\mathrm{C}u=1-2q\cos\pi u+2q^4\cos 2\pi u-2q^9\cos 3\pi u+\ldots\ (=\Theta(\mathrm{K}u),\ \text{Jacobi}),$$

$$\vartheta{}^{1}_{1}(u)=\vartheta_3 u=\mathrm{D}u=-2q^{\frac{1}{4}}\sin\tfrac{1}{2}\pi u+2q^{\frac{9}{4}}\sin\tfrac{3}{2}\pi u-2q^{\frac{25}{4}}\cos\tfrac{5}{2}\pi u+\ldots\ (=-\mathrm{H}(\mathrm{K}u),\ \text{Jacobi}),$$

where a is of the form $a=-\alpha+\beta i$, α being non-evanescent and positive: hence $q=\exp.(-\alpha+\beta i)=e^{-\alpha}(\cos\beta+i\sin\beta)$, where $e^{-\alpha}$, the modulus of q is positive and less than 1; $\cos\beta$ may be either positive or negative, and $q^{\frac{1}{4}}$ is written to denote exp. $\frac{1}{4}(-\alpha+\beta i)$, viz.: this is $=e^{-\frac{1}{4}\alpha}\{\cos\frac{1}{4}\beta+i\sin\frac{1}{4}\beta\}$. But usually $\beta=0$, viz., q is a real positive quantity less than 1, and $q^{\frac{1}{4}}$ denotes the real fourth root of q.

I have given above the three notations, but as already mentioned propose to employ for the four functions the notation Au, Bu, Cu, Du: it will be observed that Du is an odd function, but that Au, Bu, Cu are even functions of u.

The constants of the theory.

29. We have

$$\begin{aligned}
\mathrm{A}0&=1+2q+2q^4+2q^9+\ldots\quad,\\
\mathrm{B}0&=\phantom{1+{}}2q^{\frac{1}{4}}+2q^{\frac{9}{4}}+2q^{\frac{25}{4}}+\ldots\quad,\\
\mathrm{C}0&=1-2q+2q^4-2q^9+\ldots\quad,\\
\mathrm{D}0&=0\quad,\\
\mathrm{D}'0&=-\pi\{q^{\frac{1}{4}}-3q^{\frac{9}{4}}=5q^{\frac{25}{4}}-\ldots\}.
\end{aligned}$$

If, as definitions of k, k', K, we assume

$$k=\frac{\mathrm{B}^2 0}{\mathrm{A}^2 0},\ k'=\frac{\mathrm{C}^2 0}{\mathrm{A}^2 0},\ \mathrm{K}=-\frac{\mathrm{A}0}{\mathrm{B}0}\cdot\frac{\mathrm{D}'0}{\mathrm{C}0},$$

then we have

$$\begin{aligned}
k&=4\sqrt{q}\left\{\frac{1+q^2+q^6+\ldots}{1+2q+2q^4+\ldots}\right\}^2,\ =4\sqrt{q}(1-4q+14q^2+\ldots),\\
k'&=\phantom{4\sqrt{q}}\left\{\frac{1-2q+2q^4-\ldots}{1+2q+2q^4+\ldots}\right\}^2,\ =1-8q+32q^2-96q^3+\ldots,\\
\mathrm{K}&=\frac{\pi(1+2q+2q^4+\ldots)(1-3q^2+5q^6-\ldots)}{2(1-2q+2q^4-\ldots)(1+q^2+q^6+\ldots)},\ =\tfrac{1}{2}\pi(1+4q+4q^2+0q^3+\ldots),
\end{aligned}$$

where I have added the first few terms of the expansions of these quantities. We have identically

$$k^2+k'^2=1.$$

It will be convenient to write also as the definition of E,

$$\mathrm{K}(\mathrm{K}-\mathrm{E})=\frac{\mathrm{C}''0}{\mathrm{C}0}:$$

we have then

$$\mathrm{E}=\mathrm{K}-\frac{1}{\mathrm{K}}\frac{\mathrm{C}''0}{\mathrm{C}0},\ =\frac{1}{\mathrm{A}0.\mathrm{B}0.\mathrm{C}0.\mathrm{D}'0}\{-\mathrm{A}^2 0(\mathrm{D}'0)^2+\mathrm{B}^2 0.\mathrm{C}0.\mathrm{C}''0\},$$

and moreover

$$1-\frac{\mathrm{E}}{\mathrm{K}}=\frac{1}{\mathrm{K}^2}\cdot\frac{\mathrm{C}''0}{\mathrm{C}0},\ =\frac{2\pi^2}{\mathrm{K}^2}\cdot\frac{q-4q^4+9q^9-\ldots}{1-2q+2q^4+\ldots},$$

giving

$$\frac{\mathrm{E}}{\mathrm{K}} = 1-8q+48q^2-224q^3+\ldots,$$

and thence

$$\mathrm{E}=\tfrac{1}{2}\pi\{1-4q+20q^2-64q^3+\ldots\}.$$

30. Other formulæ are

$$k=4\sqrt{q}\left\{\frac{1+q^2.1+q^4\ldots}{1+q.1+q^3\ldots}\right\}^4,$$

$$k'=\left\{\frac{1-q.1-q^3\ldots}{1+q.1+q^3\ldots}\right\}^4,$$

$$\mathrm{K}=\tfrac{1}{2}\pi\left\{\frac{1+q.1+q^3\ldots1-q^2.1-q^4\ldots}{1-q.1-q^3\ldots1+q^2.1+q^4\ldots}\right\}^2.$$

31. Jacobi's definition of q is from a different point of view altogether, viz., we have $q=\exp.-\frac{\pi\mathrm{K}'}{\mathrm{K}}$, where

$$\mathrm{K}=\int_0^1\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}};$$

and K' is the like function of k'; the equation gives $\log q=-\frac{\pi\mathrm{K}'}{\mathrm{K}}$; viz., we have

$$\mathrm{K}'=-\frac{\mathrm{K}}{\pi}\log q,$$

and, regarding herein K as a given function of q, this equation gives K' as a function of q.

The product-theorem.

32. The product-theorem is

$$\vartheta\begin{pmatrix}\alpha\\\gamma\end{pmatrix}(u+u').\vartheta\begin{pmatrix}\alpha'\\\gamma'\end{pmatrix}(u-u')=\Theta\begin{pmatrix}\tfrac{1}{2}(\alpha+\alpha')\\\gamma+\gamma'\end{pmatrix}2u.\Theta\begin{pmatrix}\tfrac{1}{2}(\alpha-\alpha')\\\gamma-\gamma'\end{pmatrix}2u'+\Theta\begin{pmatrix}\tfrac{1}{2}(\alpha+\alpha')+1\\\gamma+\gamma'\end{pmatrix}2u.\Theta\begin{pmatrix}\tfrac{1}{2}(\alpha-\alpha')+1\\\gamma-\gamma'\end{pmatrix}2u',$$

and here giving to $\begin{matrix}\alpha\\\gamma\end{matrix}, \begin{matrix}\alpha'\\\gamma'\end{matrix}$ their different values, and introducing unaccented and accented capitals to denote the functions of $2u$ and $2u'$ respectively, the 16 equations are

A.A $\vartheta^{0}_{0}u+u'\vartheta^{0}_{0}u-u'=\mathrm{XX'}+\mathrm{YY'}$, (square-set)

B.B ϑ^{1}_{0} ,, ϑ^{1}_{0} ,, $=\mathrm{YX'}+\mathrm{XY'}$,

C.C ϑ^{0}_{1} ,, ϑ^{0}_{1} ,, $=\mathrm{XX'}-\mathrm{YY'}$,

D.D ϑ^{1}_{1} ,, ϑ^{1}_{1} ,, $=-\mathrm{YX'}+\mathrm{XY'}$;

C.A $\vartheta^{0}_{1}u+u'\vartheta^{0}_{0}u-u'=\mathrm{X_{,}X_{,}'}+\mathrm{Y_{,}Y_{,}'}$, (first product-set)

A.C ϑ^{0}_{0} ,, ϑ^{0}_{1} ,, $=\mathrm{X_{,}X_{,}'}-\mathrm{Y_{,}Y_{,}'}$,

D.B ϑ^{1}_{1} ,, ϑ^{1}_{0} ,, $=\mathrm{Y_{,}X_{,}'}+\mathrm{X_{,}Y_{,}'}$,

B.D ϑ^{1}_{0} ,, ϑ^{1}_{1} ,, $=\mathrm{Y_{,}X_{,}'}-\mathrm{X_{,}Y_{,}'}$;

B.A $\vartheta^{1}_{0}u+u'\vartheta^{0}_{0}u-u'=\mathrm{PP'}+\mathrm{QQ'}$, (second product-set)

A.B ϑ^{0}_{0} ,, ϑ^{1}_{0} ,, $=\mathrm{PQ'}+\mathrm{QP'}$,

D.C ϑ^{1}_{1} ,, ϑ^{0}_{1} ,, $=i\mathrm{PP'}-i\mathrm{QQ'}$,

C.D ϑ^{0}_{1} ,, ϑ^{1}_{1} ,, $=i\mathrm{PQ'}-i\mathrm{QP'}$;

D.A $\vartheta^{1}_{1}u+u'\vartheta^{0}_{0}u-u'=\mathrm{P_{,}P_{,}'}+\mathrm{Q_{,}Q_{,}'}$, (third product-set)

A.D ϑ^{0}_{0} ,, ϑ^{1}_{1} ,, $=i\mathrm{P_{,}Q_{,}'}-i\mathrm{Q_{,}P_{,}'}$,

B.C ϑ^{1}_{0} ,, ϑ^{0}_{1} ,, $=-i\mathrm{P_{,}P_{,}'}+i\mathrm{Q_{,}Q_{,}'}$,

C.B ϑ^{0}_{1} ,, ϑ^{1}_{0} ,, $=\mathrm{P_{,}Q_{,}'}+\mathrm{Q_{,}P_{,}'}$.

33. Here, and subsequently, we have

$\Theta^{0}_{0}, \Theta^{1}_{0}, \Theta^{0}_{1}, \Theta^{1}_{1}\,(2u)=\mathrm{X},\ \mathrm{Y},\ \mathrm{X}_{,},\ \mathrm{Y}_{,}$
,, ,, ,, ,, $(2u')=\mathrm{X}',\ \mathrm{Y}',\ \mathrm{X}_{,}',\ \mathrm{Y}_{,}'$
,, ,, ,, ,, $(0)=\alpha,\ \beta,\ \alpha_{,},\ \beta_{,}$

$\Theta^{\frac{1}{2}}_{0}, \Theta^{\frac{3}{2}}_{0}, \Theta^{\frac{1}{2}}_{1}, \Theta^{\frac{3}{2}}_{1}\,(2u)=\mathrm{P},\ \mathrm{Q},\ \mathrm{P}_{,},\ \mathrm{Q}_{,}$
,, ,, ,, ,, $(2u')=\mathrm{P}',\ \mathrm{Q}',\ \mathrm{P}_{,}',\ \mathrm{Q}_{,}'$
,, ,, ,, ,, $(0)=\mathrm{p},\ \mathrm{q},\ \mathrm{p}_{,},\ \mathrm{q}_{,}$

viz., we use also $\alpha, \beta, \alpha_{,}, \beta_{,}$ and $\mathrm{p}, \mathrm{q}, \mathrm{p}_{,}, \mathrm{q}_{,}$ to denote the zero-functions; $\beta_{,}$ is $=0$, but we use $\beta_{,}'$ to denote the zero-value of $\frac{d}{du}\mathrm{Y}_{,}$.

34. For obtaining the foregoing relations it is necessary to observe that

$$\Theta^{\alpha+2}_{\gamma} = \Theta^{\alpha}_{\gamma};$$

by which the upper character is always reduced to 0, 1, $\frac{1}{2}$ or $\frac{3}{2}$; and that for reducing the lower character we have

$$\Theta^{0}_{\gamma+2} = \Theta^{0}_{\gamma};\ \Theta^{1}_{\gamma+2} = -\Theta^{1}_{\gamma};$$

$$\Theta^{\frac{1}{2}}_{\gamma+2} = i\Theta^{\frac{1}{2}}_{\gamma},\ \Theta^{\frac{1}{2}}_{\gamma-2} = -i\Theta^{\frac{1}{2}}_{\gamma};\ \Theta^{\frac{3}{2}}_{\gamma+2} = -i\Theta^{\frac{3}{2}}_{\gamma},\ \Theta^{\frac{3}{2}}_{\gamma-2} = i\Theta^{\frac{3}{2}}_{\gamma};$$

by means of which the lower character is always reduced to 0 or 1: in all these formulæ the argument is arbitrary, and it is thus $=2u$, or $2u'$ as the case requires. The formulæ are obtained without difficulty directly from the definition of the functions Θ.

35. As an instance, taking $\frac{\alpha}{\gamma}, \frac{\alpha'}{\gamma'} = \frac{1}{1}, \frac{0}{1}$, the product-equation is

$$\begin{aligned}\vartheta^{1}_{1}(u+u').\vartheta^{0}_{1}(u-u') &= \Theta^{\frac{1}{2}}_{2}(2u).\Theta^{\frac{1}{2}}_{0}(2u') + \Theta^{\frac{3}{2}}_{2}(2u).\Theta^{\frac{3}{2}}_{0}(2u'),\\ &= i\Theta^{\frac{1}{2}}_{0}(2u).\Theta^{\frac{1}{2}}_{0}(2u') - i\Theta^{\frac{3}{2}}_{0}(2u).\Theta^{\frac{3}{2}}_{0}(2u'),\\ &= i\mathrm{P}.\mathrm{P}' \qquad\qquad\qquad - i\mathrm{Q}.\mathrm{Q}',\end{aligned}$$

which agrees with the before-given value.

36. The following values are not actually required, but I give them to fix the ideas, and show the meaning of the quantities with which we work.

$$\begin{array}{l|l} & u=0 \\ \mathrm{X} = \Theta^{0}_{0}(2u) = 1 + 2q^{2}\cos 2\pi u + 2q^{8}\cos 4\pi u + \ldots, & \alpha = 1 + 2q^{2} + 2q^{8} + \ldots, \\ \mathrm{Y} = \Theta^{1}_{0}(2u) = \quad 2q^{\frac{1}{2}}\cos \pi u + 2q^{\frac{9}{2}}\cos 3\pi u + \ldots, & \beta = \quad 2q^{\frac{1}{2}} + 2q^{\frac{9}{2}} + \ldots, \\ \mathrm{X}_{,} = \Theta^{0}_{1}(2u) = 1 - 2q^{2}\cos 2\pi u + 2q^{8}\cos 4\pi u + \ldots, & \alpha_{,} = 1 - 2q^{2} + 2q^{8} \ldots, \\ \mathrm{Y}_{,} = \Theta^{1}_{1}(2u) = -2q^{\frac{1}{2}}\sin \pi u + 2q^{\frac{9}{2}}\sin 3\pi u + \ldots, & \beta_{,}' = 2\pi(-q^{\frac{1}{2}} + 3q^{\frac{3}{2}} - \ldots) \end{array}$$

$$= \frac{d}{du}\mathrm{Y}_{,} \text{ for } u=0.$$

$$\begin{aligned}
\mathrm{P} &= \Theta_0^{\frac{1}{2}}(2u) = q^{\frac{1}{8}}(\cos\tfrac{1}{2}\pi u + i\sin\tfrac{1}{2}\pi u) + q^{\frac{9}{8}}(\cos\tfrac{3}{2}\pi u - i\sin\tfrac{3}{2}\pi u) \\
&\qquad + q^{\frac{25}{8}}(\cos\tfrac{5}{2}\pi u + i\sin\tfrac{5}{2}\pi u) + \ldots,
\end{aligned}$$

$$\begin{aligned}
\mathrm{Q} &= \Theta_0^{\frac{3}{2}}(2u) = q^{\frac{1}{8}}(\cos\tfrac{1}{2}\pi u - i\sin\tfrac{1}{2}\pi u) + q^{\frac{9}{8}}(\cos\tfrac{3}{2}\pi u + i\sin\tfrac{3}{2}\pi u) \\
&\qquad + q^{\frac{25}{8}}(\cos\tfrac{5}{2}\pi u - i\sin\tfrac{5}{2}\pi u) + \ldots,
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}_{,} &= \Theta_1^{\frac{1}{2}}(2u) = \frac{1+i}{\sqrt{2}}\Big\{ q^{\frac{1}{8}}(\cos\tfrac{1}{2}\pi u + i\sin\tfrac{1}{2}\pi u) - q^{\frac{9}{8}}(\cos\tfrac{3}{2}\pi u - i\sin\tfrac{3}{2}\pi u) \\
&\qquad - q^{\frac{25}{8}}(\cos\tfrac{5}{2}\pi u + i\sin\tfrac{5}{2}\pi u) + + \ldots,
\end{aligned}$$

$$\begin{aligned}
\mathrm{Q}_{,} &= \Theta_1^{\frac{3}{2}}(2u) = \frac{1-i}{\sqrt{2}}\Big\{ q^{\frac{1}{8}}(\cos\tfrac{1}{2}\pi i - i\sin\tfrac{1}{2}\pi u) - q^{\frac{9}{8}}(\cos\tfrac{3}{2}\pi u + i\sin\tfrac{3}{2}\pi u) \\
&\qquad - q^{\frac{25}{8}}(\cos\tfrac{5}{2}\pi u - i\sin\tfrac{5}{2}\pi u) + + \ldots;
\end{aligned}$$

and therefore also

$$\mathrm{p} = \mathrm{q} = q^{\frac{1}{8}} + q^{\frac{9}{8}} + q^{\frac{25}{8}} \ldots,$$

$$\mathrm{p}_{,} = \frac{1+i}{\sqrt{2}}\left\{ q^{\frac{1}{8}} - q^{\frac{9}{8}} - q^{\frac{25}{8}} + q^{\frac{49}{8}} + q^{\frac{81}{8}} - - \ldots \right\},\ \mathrm{q}_{,} = \frac{1-i}{\sqrt{2}}\left\{\text{Do.}\right\};\ \mathrm{p}_{,} = i\mathrm{q}_{,}$$

The square set, $u'=0$; and x-formulæ.

37. We use the square-set, in the first instance by writing therein $u'=0$; the equations become

$$\begin{aligned}
\mathrm{A}^2u &= \alpha\mathrm{X} + \beta\mathrm{Y},\ = \omega^2\mathfrak{A}(a-x), \\
\mathrm{B}^2u &= \beta\mathrm{X} + \alpha\mathrm{Y},\ = \omega^2\mathfrak{B}(b-x), \\
\mathrm{C}^2u &= \alpha\mathrm{X} - \beta\mathrm{Y},\ = \omega^2\mathfrak{C}(c-x), \\
\mathrm{D}^2u &= \beta\mathrm{X} - \alpha\mathrm{Y},\ = \omega^2\mathfrak{D}(d-x),
\end{aligned}$$

viz., the equations without their last members show that there exist functions ω^2 and $x\omega^2$, linear functions of X and Y, such that $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{A}a$, $\mathfrak{B}b$, $\mathfrak{C}c$, $\mathfrak{D}d$, being constants, the squared functions may be assumed equal to $\mathfrak{A}a.\omega^2 - \mathfrak{A}.\omega^2x$, &c., that is, $\omega^2\mathfrak{A}(a-x)$, &c., respectively: the squared functions are then *proportional* to the values $\mathfrak{A}(a-x)$, &c.

To show the meaning of the factor ω^2, observe that from any two of the equations, for instance the first and second, we have an equation without ω, $\mathrm{A}^2u \div \mathrm{B}^2u = \mathfrak{A}(a-x) \div \mathfrak{B}(b-x)$; and using this to determine x, and then substituting in $\omega^2 = \mathrm{A}^2u \div \mathfrak{A}(a-x)$, we find

$$\omega^2 = \frac{\mathfrak{B}\mathrm{A}^2u - \mathfrak{A}\mathrm{B}^2u}{(a-b)\mathfrak{A}\mathfrak{B}},$$

where the numerator is a function not in anywise more important than any other linear function of A^2u and B^2u.

38. The function Du vanishes for $u=0$, and we may assume that the corresponding value of x is $=d$. Writing in the other equations $u=0$, they become

$$\begin{aligned} A^2 0 &= (\alpha^2+\beta^2) = \omega_0{}^2\mathfrak{A}(a-d), \\ B^2 0 &= \quad 2\alpha\beta \quad = \omega_0{}^2\mathfrak{B}(b-d), \\ C^2 0 &= \ \alpha^2-\beta^2 \ = \omega_0{}^2\mathfrak{C}(c-d), \end{aligned}$$

where $\omega_0{}^2$ is what ω^2 becomes on writing therein $x=d$. It is convenient to omit altogether these factors ω^2 and $\omega_0{}^2$; it being understood that without them, the equations denote not absolute equalities, but only equalities of ratios: thus, without the $\omega_0{}^2$, the last-mentioned equations would denote $A^2 0 : B^2 0 : C^2 0 = \alpha^2+\beta^2 : 2\alpha\beta : \alpha^2-\beta^2$, $=\mathfrak{A}(a-d) : \mathfrak{B}(b-d) : \mathfrak{C}(c-d)$. The quantities $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ only present themselves in the products $\mathfrak{A}\omega^2$, &c., and their absolute magnitudes are therefore essentially indeterminate, but regarding ω^2 as containing a constant factor of properly determined value, the absolute values of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ may be regarded as determinate, and this is accordingly done in the formulæ $\mathfrak{A}^2=-\mathrm{agh}$, &c., which follow.

Relations between the constants.

39. The formulæ contain the differences of the quantities a, b, c, d; denoting these differences in the usual manner

$$b-c,\ c-a,\ a-b,\ a-d,\ b-d,\ c-d$$

by

$$\mathrm{a},\quad \mathrm{b},\quad \mathrm{c},\quad \mathrm{f},\quad \mathrm{g},\quad \mathrm{h}$$

so that

$$\begin{array}{rrrrl} . & -\mathrm{h} & +\mathrm{g} & -\mathrm{a} & =0, \\ \mathrm{h} & . & -\mathrm{f} & -\mathrm{b} & =0, \\ -\mathrm{g} & +\mathrm{f} & . & -\mathrm{c} & =0, \\ \mathrm{a} & +\mathrm{b} & +\mathrm{c} & . & =0, \end{array}$$

and also

$$\mathrm{af}+\mathrm{bg}+\mathrm{ch} \qquad =0,$$

and then assuming the absolute value of one of the quantities $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, we have the system of relations

$$\begin{array}{llll}
\mathfrak{A}^2=-\mathrm{agh}, & \mathfrak{B}\mathfrak{C}\mathrm{a}=\ \mathfrak{A}\mathfrak{D}\mathrm{f}, & \mathfrak{A}\mathrm{bcf}=-\mathfrak{B}\mathfrak{C}\mathfrak{D}, & \mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D}=\mathrm{abcfgh}, \\
\mathfrak{B}^2=\ \ \mathrm{bhf}, & \mathfrak{C}\mathfrak{A}\mathrm{b}=-\mathfrak{B}\mathfrak{D}\mathrm{g}, & \mathfrak{B}\mathrm{cag}=\ \ \mathfrak{C}\mathfrak{A}\mathfrak{D}, & \\
\mathfrak{C}^2=\ \ \mathrm{cfg}, & \mathfrak{A}\mathfrak{B}\mathrm{c}=-\mathfrak{C}\mathfrak{D}\mathrm{h}, & \mathfrak{C}\mathrm{abh}=\ \ \mathfrak{A}\mathfrak{B}\mathfrak{D}, & \\
\mathfrak{D}^2=-\mathrm{abc}, & & \mathfrak{D}\mathrm{fgh}=-\mathfrak{A}\mathfrak{B}\mathfrak{C}, &
\end{array}$$

$$\begin{array}{rrrrl}
. & \mathrm{c}^2\mathfrak{B}^2 & +\mathrm{b}^2\mathfrak{C}^2 & -\mathrm{f}^2\mathfrak{D}^2 & =\mathrm{bcf}\,(\mathrm{af}+\mathrm{bg}+\mathrm{ch}),\ =0, \\
-\mathrm{c}^2\mathfrak{A}^2 & . & +\mathrm{a}^2\mathfrak{C}^2 & -\mathrm{g}^2\mathfrak{D}^2 & =\mathrm{cag}(\quad ,, \quad),\ =0, \\
-\mathrm{b}^2\mathfrak{A}^2 & +\mathrm{a}^2\mathfrak{B}^2 & . & -\mathrm{h}^2\mathfrak{D}^2 & =\mathrm{abh}(\quad ,, \quad),\ =0, \\
-\mathrm{f}^2\mathfrak{A}^2 & +\mathrm{g}^2\mathfrak{B}^2 & +\mathrm{h}^2\mathfrak{C}^2 & . & =\mathrm{fgh}\,(\quad ,, \quad),\ =0.
\end{array}$$

It is to be remarked that taking c, a, b, d in the order of decreasing magnitude we have $-\mathrm{a}$, b, c, f, g, h all positive; hence $\mathfrak{A}^2$, $\mathfrak{B}^2$ $\mathfrak{C}^2$, $\mathfrak{D}^2$ all real; and taking as we may do, $\mathfrak{D}$ negative, then $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ may be taken positive; that is we have $-\mathrm{a}$, b, c, f, g, h, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $-\mathfrak{D}$ all of them positive.

40. We have

$$\begin{aligned}
\mathrm{A}^2 0 &= \alpha^2+\beta^2=\mathfrak{A}\mathrm{f}, \\
\mathrm{B}^2 0 &= \ 2\alpha\beta\ =\mathfrak{B}\mathrm{g}, \\
\mathrm{D}^2 0 &= \alpha^2-\beta^2=\mathfrak{C}\mathrm{h}.
\end{aligned}$$

The foregoing equations

$$k=\frac{\mathrm{B}^2 0}{\mathrm{A}^2 0}, \qquad k'=\frac{\mathrm{C}^2 0}{\mathrm{A}^2 0},$$

give

$$k=\frac{\mathfrak{B}\mathrm{g}}{\mathfrak{A}\mathrm{f}}, \qquad k'=\frac{\mathfrak{C}\mathrm{h}}{\mathfrak{A}\mathrm{f}},$$

and we thence have

$$k^2=\frac{\mathrm{bg}}{-\mathrm{af}}, \qquad k'^2=\frac{\mathrm{ch}}{-\mathrm{af}}, \quad \text{satisfying } k^2+k'^2=1.$$

41. Observe further that substituting for a, b, c, f, g, h their values, we have

$$\begin{array}{lcl}
\mathfrak{A}^2=c-b.b-d.c-d, & = & c-d.d-b.b-c, \\
\mathfrak{B}^2=c-a.c-d.a-d, & = & d-a.a-c.c-d, \\
\mathfrak{C}^2=a-b.a-d.b-d, & = & -.a-b.b-d.d-a, \\
\mathfrak{D}^2=c-b.c-a.a-b, & = & -.b-c.c-a.a-b,
\end{array}$$

where in the first set of values all the differences are positive, but in the second set of values, we take the triads of *abcd,* in the cyclical order *bcd, cda, abc.* There is in this last form an apparent want of symmetry as to the signs (viz. : the order which might have been expected is $+-+-$), but taking the order of the letters to be $\mathfrak{C},\mathfrak{A},\mathfrak{B},\mathfrak{D}$ and $c,a,b,d,$ then the cyclical arrangement is

$$\begin{aligned}
\mathfrak{C}^2 &= -.b-d.d-a.a-b.\\
\mathfrak{A}^2 &= -.d-c.c-b.b-d.\\
\mathfrak{B}^2 &= -.c-a.a-d.d-c.\\
\mathfrak{D}^2 &= -.a-b.b-c.c-a.
\end{aligned}$$

where the four outside signs are all $-$. Observe that the triads of *abcd,* and *abdc,* are

$$\begin{array}{lcccc}
 & bcd, & cda, & dab, & abc,\\
\text{and} & bdc, & dca, & cab, & abd,
\end{array}$$

where in the first and second columns the terms of the same column correspond to each other with a reversal of sign, whereas in the third and fourth columns the lower term of either column corresponds to the upper term of the other column, but without a reversal of sign.

The product-sets, $u\pm u'$: and u' indefinitely small, differential formulæ.

42. Coming now to the product-sets, these may be written

$$\begin{array}{l|l}
\tfrac{1}{2}\{\overset{u+u'}{\mathrm{C}}.\overset{u-u'}{\mathrm{A}} + \overset{u+u'}{\mathrm{A}}.\overset{u-u'}{\mathrm{C}}\}=\mathrm{X}_{,}\mathrm{X}_{,}', & \tfrac{1}{2}\{\overset{u+u'}{\mathrm{C}}.\overset{u-u'}{\mathrm{A}} - \overset{u+u'}{\mathrm{A}}.\overset{u-u'}{\mathrm{C}}\}=\mathrm{Y}_{,}\mathrm{Y}_{,}',\\
\;,,\{\mathrm{D.B} + \mathrm{B.D}\}=\mathrm{Y}_{,}\mathrm{X}_{,}', & \;,,\{\mathrm{D.B} - \mathrm{B.D}\}=\mathrm{X}_{,}\mathrm{Y}_{,}',\\
 & \\
\tfrac{1}{2}\{\mathrm{B.A} + \mathrm{A.B}\}=\quad(\mathrm{P}+\mathrm{Q})(\mathrm{P}'+\mathrm{Q}'), & \tfrac{1}{2}\{\mathrm{B.A} - \mathrm{A.B}\}=\quad(\mathrm{P}-\mathrm{Q})(\mathrm{P}'-\mathrm{Q}'),\\
\;,,\{\mathrm{D.C} + \mathrm{C.D}\}=\quad i(\mathrm{P}-\mathrm{Q})(\mathrm{P}'+\mathrm{Q}'), & \;,,\{\mathrm{D.C} - \mathrm{C.D}\}=\quad i(\mathrm{P}+\mathrm{Q})(\mathrm{P}'-\mathrm{Q}'),\\
 & \\
\tfrac{1}{2}\{\mathrm{D.A} + \mathrm{A.D}\}=\quad(\mathrm{P}_{,}-i\mathrm{Q}_{,})(\mathrm{P}_{,}'+i\mathrm{Q}_{,}'), & \tfrac{1}{2}\{\mathrm{D.A} - \mathrm{A.D}\}=\quad(\mathrm{P}_{,}+i\mathrm{Q}_{,})(\mathrm{P}_{,}'-i\mathrm{Q}_{,}'),\\
\;,,\{\mathrm{B.C} + \mathrm{C.B}\}=-i(\mathrm{P}_{,}-i\mathrm{Q}_{,})(\mathrm{P}_{,}'+i\mathrm{Q}_{,}'), & \;,,\{\mathrm{B.C} - \mathrm{C.B}\}=-i(\mathrm{P}_{,}-i\mathrm{Q}_{,})(\mathrm{P}_{,}'-i\mathrm{Q}_{,}').
\end{array}$$

43. We can from each set form two fractions (each of them a function of $u+u'$ and $u-u'$), which are equal to one and the same function of u' only: for instance, from the first set we have two fractions, each $\frac{\mathrm{Y}_{,}'}{\mathrm{X}'}$: putting in such equation $u=0$, we obtain a new expression for the function of u' involving only the theta-functions $\mathrm{A}u'$, &c.,

which new expression we may then substitute in the equations first obtained: we thus arrive at the six equations

$$\frac{\overset{u+u'}{\mathrm{C}}\,.\,\overset{u-u'}{\mathrm{A}}-\overset{u+u'}{\mathrm{A}}\,.\,\overset{u-u'}{\mathrm{C}}}{\mathrm{D}\,.\,\mathrm{B}+\mathrm{B}\,.\,\mathrm{D}}=\frac{\overset{u+u'}{\mathrm{D}}\,.\,\overset{u-u'}{\mathrm{B}}-\overset{u+u'}{\mathrm{B}}\,.\,\overset{u-u'}{\mathrm{D}}}{\mathrm{C}\,.\,\mathrm{A}+\mathrm{A}\,.\,\mathrm{C}}=\frac{\mathrm{D}u'.\mathrm{B}u'}{\mathrm{C}u'.\mathrm{A}u'},$$

$$-\frac{\mathrm{B}\,.\,\mathrm{A}-\mathrm{A}\,.\,\mathrm{B}}{\mathrm{D}\,.\,\mathrm{C}+\mathrm{C}\,.\,\mathrm{D}}=\frac{\mathrm{D}\,.\,\mathrm{C}-\mathrm{C}\,.\,\mathrm{D}}{\mathrm{B}\,.\,\mathrm{A}+\mathrm{A}\,.\,\mathrm{B}}=\frac{\mathrm{D}u'.\mathrm{C}u'}{\mathrm{B}u'.\mathrm{A}u'},$$

$$-\frac{\mathrm{B}\,.\,\mathrm{C}-\mathrm{C}\,.\,\mathrm{B}}{\mathrm{D}\,.\,\mathrm{A}+\mathrm{A}\,.\,\mathrm{D}}=\frac{\mathrm{D}\,.\,\mathrm{A}-\mathrm{A}\,.\,\mathrm{D}}{\mathrm{B}\,.\,\mathrm{C}+\mathrm{C}\,.\,\mathrm{B}}=\frac{\mathrm{D}u'.\mathrm{A}u'}{\mathrm{B}u'.\mathrm{C}u'},$$

where observe that the expressions all vanish for $u'=0$.

44. Taking herein u' indefinitely small we obtain

$$\frac{\mathrm{A}u.\mathrm{C}'u-\mathrm{C}u\mathrm{A}'u}{\mathrm{B}u.\mathrm{D}u}=\frac{\mathrm{B}u.\mathrm{D}'u-\mathrm{D}u.\mathrm{B}'u}{\mathrm{C}u.\mathrm{A}u}=\frac{\mathrm{D}'0.\mathrm{B}0}{\mathrm{C}'0.\mathrm{A}0}=-\mathrm{K}\frac{\mathrm{B}^2 0}{\mathrm{A}^2 0},$$

$$-\frac{\mathrm{A}u.\mathrm{B}'u-\mathrm{B}u\mathrm{A}'u}{\mathrm{C}u.\mathrm{D}u}=\frac{\mathrm{C}u.\mathrm{D}'u-\mathrm{D}u.\mathrm{C}'u}{\mathrm{C}u.\mathrm{B}u}=\frac{\mathrm{D}'0.\mathrm{C}0}{\mathrm{A}0.\mathrm{B}0}=-\mathrm{K}\frac{\mathrm{C}^2 0}{\mathrm{A}^2 0},$$

$$-\frac{\mathrm{C}u.\mathrm{B}'u-\mathrm{B}u.\mathrm{C}'u}{\mathrm{A}u.\mathrm{D}u}=\frac{\mathrm{A}u.\mathrm{D}'u-\mathrm{D}u.\mathrm{A}'u}{\mathrm{B}u\,\mathrm{C}u}=\frac{\mathrm{D}'0.\mathrm{A}0}{\mathrm{B}0.\mathrm{C}0}=-\mathrm{K},$$

where the last column is added in order to introduce K in place of D′0.

45. These formulæ in effect give the derivatives of the quotient-functions in terms of quotient-functions: for instance, one of the equations is

$$\frac{d}{du}\frac{\mathrm{D}u}{\mathrm{A}u}=-\mathrm{K}\frac{\mathrm{B}u}{\mathrm{A}u}\cdot\frac{\mathrm{C}u}{\mathrm{A}u};$$

substituting herein for the quotient-fractions their values in terms of x, this becomes

$$\frac{d}{du}\sqrt{\frac{d-x}{a-x}}=-\mathrm{K}\sqrt{\frac{\mathfrak{BC}}{\mathfrak{AD}}}\,\frac{\sqrt{b-x.c-x}}{a-x},\ =-\mathrm{K}\sqrt{\frac{\mathrm{f}}{\mathrm{a}}}\,\frac{\sqrt{b-x.c-x}}{a-x},$$

or the left hand being $=\dfrac{-\frac{1}{2}\mathrm{f}}{(a-x)^{\frac{3}{2}}\sqrt{d-x}}\dfrac{dx}{du}$, this is

$$\mathrm{K}du=\frac{\frac{1}{2}\sqrt{\mathrm{af}}.dx}{\sqrt{a-x.b-x.c-x.d-x}},$$

where on the right hand side it would be better to write $\sqrt{-\mathrm{af}}$ in the numerator, and $x-d$ in place of $d-x$ in the denominator.

Comparison with JACOBI.

46. The comparison of the formulæ with JACOBI gives

$$\begin{aligned}
\mathrm{sn}\,\mathrm{K}u &= -\frac{1}{\sqrt{k}}\,\mathrm{D}u \div \mathrm{C}u, &&= \sqrt{\frac{\mathrm{a}}{\mathrm{g}}}\sqrt{\frac{d-x}{c-x}}\left(\text{or better } \sqrt{\frac{-\mathrm{a}}{\mathrm{g}}}\sqrt{\frac{x-d}{c-x}}\right),\\
\mathrm{cn}\,\mathrm{K}u &= \sqrt{\frac{k'}{k}}\,\mathrm{B}u \div \mathrm{C}u, &&= \sqrt{\frac{\mathrm{h}}{\mathrm{g}}}\sqrt{\frac{b-x}{c-x}},\\
\mathrm{dn}\,\mathrm{K}u &= \sqrt{k'}\,\mathrm{A}u \div \mathrm{C}u, &&= \sqrt{\frac{\mathrm{h}}{\mathrm{f}}}\sqrt{\frac{a-x}{c-x}},
\end{aligned}$$

where it will be recollected that

$$k^2 = \frac{\mathrm{bg}}{-\mathrm{af}},\quad k'^2 = \frac{\mathrm{ch}}{-\mathrm{af}}.$$

It may be remarked that we seek to determine everything in terms of a, b, c, d. The formula just written down, $k^2 = \mathrm{bg} \div -\mathrm{af}$, gives k in terms of these quantities; and k, K being each given in terms of q, we have virtually K as a function of k, that is of a, b, c, d: but it would not be easy from the expressions of k, K each in terms of q, to deduce the actual expression $\mathrm{K} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$, of K as a function of k.

The square-set, $u \pm u'$.

47. Reverting to the square-set

$$\begin{aligned}
\mathrm{A}(u+u')\mathrm{A}(u-u') &= \mathrm{XX}'+\mathrm{YY}',\\
\mathrm{B}(u+u')\mathrm{B}(u-u') &= \mathrm{YX}'+\mathrm{XY}',\\
\mathrm{C}(u+u')\mathrm{C}(u-u') &= \mathrm{XX}'-\mathrm{YY}',\\
\mathrm{D}(u+u')\mathrm{D}(u-u') &= -\mathrm{YX}'+\mathrm{XY}',
\end{aligned}$$

if we first write herein $u'=0$, and then $u=0$, using the results to determine the values of X, Y, X′, Y′ we find

$$\begin{array}{l|l}
\alpha\mathrm{C}^2u-\beta\mathrm{D}^2u=(\alpha^2-\beta^2)\mathrm{X}, & \alpha\mathrm{C}^2u'-\beta\mathrm{D}^2u'=(\alpha^2-\beta^2)\mathrm{X}',\\
\beta\mathrm{C}^2u-\alpha\mathrm{D}^2u=\quad ,, \quad \mathrm{Y}, & \beta\mathrm{C}^2u'-\alpha\mathrm{D}^2u'=\quad ,, \quad \mathrm{Y}',
\end{array}$$

and thence

$$\begin{aligned}(\alpha^2-\beta^2)^2\mathrm{XX}' &= \alpha^2.\mathrm{C}^2u\mathrm{C}^2u'+\beta^2.\mathrm{D}^2u\mathrm{D}^2u'-\alpha\beta(\mathrm{C}^2u\mathrm{D}^2u'+\mathrm{D}^2u\mathrm{C}^2u'),\\ \text{,,}\quad \mathrm{YY}' &= \beta^2 \quad \text{,,} \quad +\alpha^2. \quad \text{,,} \quad -\alpha\beta \qquad \text{,,} \qquad ,\end{aligned}$$

whence

$$\begin{aligned}(\alpha^2-\beta^2)^2(\mathrm{XX}'+\mathrm{YY}') &= (\alpha^2+\beta^2)(\mathrm{C}^2u\mathrm{C}^2u'+\mathrm{D}^2u\mathrm{D}^2u')-2\alpha\beta(\mathrm{C}^2u\mathrm{D}^2u'+\mathrm{D}^2u\mathrm{C}^2u'),\\ (\alpha^2-\beta^2)\ (\mathrm{XX}'-\mathrm{YY}') &= \qquad (\mathrm{C}^2u\mathrm{C}^2u'-\mathrm{D}^2u\mathrm{D}^2u'),\end{aligned}$$

(where observe that in taking the difference the right hand side becomes divisible by $\alpha^2-\beta^2$, and therefore in the final result we have on the left hand side the simple factor $\alpha^2-\beta^2$ instead of $(\alpha^2-\beta^2)^2$).

Similarly

$$\begin{aligned}(\alpha^2-\beta^2)\mathrm{YX}' &= \alpha\beta(\mathrm{C}^2u\mathrm{C}^2u'+\mathrm{D}^2u\mathrm{D}^2u')-\alpha^2\mathrm{D}^2u\mathrm{C}^2u'-\beta^2\mathrm{C}^2u\mathrm{D}^2u',\\ \text{,,}\quad \mathrm{XY}' &= \alpha\beta \qquad \text{,,} \qquad -\beta^2 \quad \text{,,} \quad -\alpha^2 \quad \text{,,} \quad ,\end{aligned}$$

and thence

$$\begin{aligned}(\alpha^2-\beta^2)^2(\ \ \mathrm{YX}'+\mathrm{XY}') &= 2\alpha\beta(\mathrm{C}^2u\mathrm{C}^2u'+\mathrm{D}^2u\mathrm{D}^2u')-(\alpha^2+\beta^2)(\mathrm{C}^2u\mathrm{D}^2u'+\mathrm{D}^2u\mathrm{C}^2u'),\\ (\alpha^2-\beta^2)\ (-\mathrm{YX}'+\mathrm{XY}') &= \quad \mathrm{D}^2u\mathrm{C}^2u'-\mathrm{C}^2u\mathrm{D}^2u',\end{aligned}$$

48. Hence recollecting that

$$\begin{aligned}\mathrm{A}^2 0 &= \alpha^2+\beta^2,\\ \mathrm{B}^2 0 &= 2\alpha\beta,\\ \mathrm{C}^2 0 &= \alpha^2-\beta^2,\end{aligned}$$

the original equations become

$$\begin{aligned}\mathrm{C}^4 0.\mathrm{A}(u+u')\mathrm{A}(u-u') &= \mathrm{A}^2 0(\mathrm{C}^2u\mathrm{C}^2u'+\mathrm{D}^2u\mathrm{D}^2u')-\mathrm{B}^2 0(\mathrm{C}^2u\mathrm{D}^2u'+\mathrm{D}^2u\mathrm{C}^2u'),\\ \mathrm{C}^4 0.\mathrm{B}(u+u')\mathrm{B}(u-u') &= \mathrm{B}^2 0(\mathrm{C}^2u\mathrm{C}^2u'+\mathrm{D}^2u\mathrm{D}^2u')-\mathrm{A}^2 0(\mathrm{C}^2u\mathrm{D}^2u'+\mathrm{D}^2u\mathrm{C}^2u'),\\ \mathrm{C}^2 0.\mathrm{C}(u+u')\mathrm{C}(u-u') &= \quad \mathrm{C}^2u\mathrm{C}^2u'-\mathrm{D}^2u\mathrm{D}^2u',\\ \mathrm{C}^2 0.\mathrm{D}(u+u')\mathrm{D}(u-u') &= \quad \mathrm{D}^2u\mathrm{C}^2u'-\mathrm{C}^2u\mathrm{D}^2u'.\end{aligned}$$

49. It will be observed that the four products $\mathrm{A}(u+u')\mathrm{A}(u-u')$, &c., are each of them expressed in terms of C^2u, D^2u, C^2u', D^2u'. Since each of the squared functions A^2u, B^2u, C^2u, D^2u is a linear function of any two of them, and the like as regards A^2u', B^2u', C^2u', D^2u', it is clear that in each equation we can on the right hand side introduce any two at pleasure of the squared functions of u, and any two at pleasure of the squared functions of u'. But all the forms so obtained are of course identical if, taking x' the same function of u' that x is of u, we introduce on the right hand side x, x' instead of u, u'; and the values of $\mathrm{A}(u+u').\mathrm{A}(u-u')$ are found to be equal to multiples of ∇, ∇_1, ∇_2, ∇_3, where

$$\nabla=x-x'\ \nabla_1=\begin{vmatrix} 1, & x+x', & xx' \\ 1, & a+d, & ad \\ 1, & b+c, & bc \end{vmatrix},\quad \nabla_2=\begin{vmatrix} 1, & x+x', & xx' \\ 1, & b+d, & bd \\ 1, & c+d, & cd \end{vmatrix},\quad \nabla_3=\begin{vmatrix} 1, & x+x', & xx' \\ 1, & c+d, & cd \\ 1, & a+b, & ab \end{vmatrix}.$$

50. In fact, from the equations $\mathrm{A}^2u=\mathfrak{A}(a-x)$, $\mathrm{A}^2u'=\mathfrak{A}(a-x')$ we have

$$\nabla=\frac{1}{\mathrm{a}\mathfrak{B}\mathfrak{C}}(\mathrm{B}^2u\mathrm{C}^2u'-\mathrm{C}^2u\mathrm{B}^2u'),\quad =\frac{1}{\mathrm{b}\mathfrak{C}\mathfrak{A}}(\mathrm{C}^2u\mathrm{A}^2u'-\mathrm{A}^2u\mathrm{C}^2u'),\quad =\frac{1}{\mathrm{c}\mathfrak{A}\mathfrak{B}}(\mathrm{A}^2u\mathrm{B}^2u'-\mathrm{B}^2u\mathrm{A}^2u'),$$

$$=\frac{1}{\mathrm{f}\mathfrak{A}\mathfrak{D}}(\mathrm{A}^2u\mathrm{D}^2u'-\mathrm{D}^2u\mathrm{A}^2u'),\quad =\frac{1}{\mathrm{g}\mathfrak{B}\mathfrak{D}}(\mathrm{B}^2u\mathrm{D}^2u'-\mathrm{D}^2u\mathrm{B}^2u'),\quad =\frac{1}{\mathrm{h}\mathfrak{C}\mathfrak{D}}(\mathrm{C}^2u\mathrm{D}^2u'-\mathrm{D}^2u\mathrm{C}^2u'),$$

where it will be recollected that $\mathrm{f}\mathfrak{A}\mathfrak{D}=\mathrm{a}\mathfrak{B}\mathfrak{C}$, $-\mathrm{g}\mathfrak{B}\mathfrak{D}=\mathrm{b}\mathfrak{C}\mathfrak{A}$, $-\mathrm{h}\mathfrak{C}\mathfrak{D}=\mathrm{c}\mathfrak{A}\mathfrak{B}$.

Moreover

$$(b-c)\nabla_1=-\begin{vmatrix} b-x.b-x', & c-x.c-x' \\ b-a.b-d, & c-a.c-d \end{vmatrix},\quad (a-d)\nabla_1=\begin{vmatrix} a-x.a-x', & d-x.d-x' \\ a-b.a-c, & d-b.d-c \end{vmatrix},$$

$$(c-a)\nabla_2=-\begin{vmatrix} c-x.c-x', & a-x.a-x' \\ c-b.c-d, & a-b.a-d \end{vmatrix},\quad (b-d)\nabla_2=\begin{vmatrix} b-x.b-x', & d-x.d-x' \\ b-c.b-a, & d-c.b-a \end{vmatrix},$$

$$(a-b)\nabla_3=-\begin{vmatrix} a-x.a-x', & b-x.b-x' \\ a-c.a-d, & b-c.b-d \end{vmatrix},\quad (c-d)\nabla_3=\begin{vmatrix} c-x.c-x', & d-x.d-x' \\ c-a.c-d, & d-a.d-b \end{vmatrix},$$

or as these may be written

$$\nabla_1=-\frac{1}{\mathrm{a}}\{\mathrm{bh}.b-x.b-x'.+\mathrm{cg}.c-x.c-x'\},\ =\frac{1}{\mathrm{f}}\{\mathrm{gh}.a-x.a-x'.+\mathrm{bc}.d-x.d-x'\},$$

$$\nabla_2=-\frac{1}{\mathrm{b}}\{\mathrm{cf}.c-x.c-x'.+\mathrm{ah}.a-x.a-x'\},\ =\frac{1}{\mathrm{g}}\{\mathrm{hf}.b-x.b-x'.+\mathrm{ca}.d-x.d-x'\},$$

$$\nabla_3=-\frac{1}{\mathrm{c}}\{\mathrm{ag}.a-x.a-x'.+\mathrm{bf}.b-x.b-x'\},=\frac{1}{\mathrm{h}}\{\mathrm{fg}.c-x.c-x'.+\mathrm{ab}.d-x.d-x'\};$$

that is

$$\nabla_1=-\frac{1}{\mathrm{a}}\left\{\frac{\mathrm{bh}}{\mathfrak{B}^2}\mathrm{B}^2u\mathrm{B}^2u'+\frac{\mathrm{cg}}{\mathfrak{C}^2}\mathrm{C}^2u\mathrm{C}^2u'\right\},\ =^{1}\left\{\frac{\mathrm{gh}}{\mathfrak{A}^2}\mathrm{A}^2u\mathrm{A}^2u'+\frac{\mathrm{bc}}{\mathfrak{D}^2}\mathrm{D}^2u\mathrm{D}^2u'\right\},$$

$$\nabla_2=-\frac{1}{\mathrm{b}}\left\{\frac{\mathrm{cf}}{\mathfrak{C}^2}\mathrm{C}^2u\mathrm{C}^2u'+\frac{\mathrm{ah}}{\mathfrak{B}^2}\mathrm{A}^2u\mathrm{A}^2u'\right\},\ =\frac{1}{\mathrm{g}}\left\{\frac{\mathrm{hf}}{\mathfrak{B}^2}\mathrm{B}^2u\mathrm{B}^2u'+\frac{\mathrm{ca}}{\mathfrak{D}^2}\mathrm{D}^2u\mathrm{D}^2u'\right\},$$

$$\nabla_3=-\frac{1}{\mathrm{c}}\left\{\frac{\mathrm{ag}}{\mathfrak{A}^2}\mathrm{A}^2u\mathrm{A}^2u'+\frac{\mathrm{bf}}{\mathfrak{C}^2}\mathrm{B}^2u\mathrm{B}^2u'\right\},\ =\frac{1}{\mathrm{h}}\left\{\frac{\mathrm{fg}}{\mathfrak{C}^2}\mathrm{C}^2u\mathrm{C}^2u'+\frac{\mathrm{ab}}{\mathfrak{D}^2}\mathrm{D}^2u\mathrm{D}^2u'\right\},$$

or finally

$$\nabla_1=-\frac{1}{\mathrm{af}}(\ \mathrm{B}^2u\mathrm{B}^2u'+\mathrm{C}^2u\mathrm{C}^2u'),=-\frac{1}{\mathrm{af}}(\mathrm{A}^2u\mathrm{A}^2u'+\mathrm{D}^2u\mathrm{D}^2u'),$$

$$\nabla_2=-\frac{1}{\mathrm{bg}}(\ \mathrm{C}^2u\mathrm{C}^2u'-\mathrm{A}^2u\mathrm{A}^2u'),=\ \frac{1}{\mathrm{bg}}(\mathrm{B}^2u\mathrm{B}^2u'-\mathrm{D}^2u\mathrm{D}^2u'),$$

$$\nabla_3=-\frac{1}{\mathrm{ch}}(-\mathrm{A}^2u\mathrm{A}^2u'+\mathrm{B}^2u\mathrm{B}^2u'),=\ \frac{1}{\mathrm{ch}}(\mathrm{C}^2u\mathrm{C}^2u'-\mathrm{D}^2u\mathrm{D}^2u').$$

51. Hence ∇, ∇_1, ∇_2, ∇_3 denoting these functions of x, x' or of u, u', we have

$$\mathrm{A}(u+u')\mathrm{A}(u-u')=\frac{\mathfrak{A}}{\mathrm{gh}}\nabla_1,$$

$$\mathrm{B}(u+u')\mathrm{B}(u-u')=\frac{\mathfrak{B}}{\mathrm{hf}}\nabla_2,$$

$$\mathrm{C}(u+u')\mathrm{C}(u-u')=\frac{\mathfrak{C}}{\mathrm{fg}}\nabla_3,$$

$$\mathrm{D}(u+u')\mathrm{D}(u-u')=\mathfrak{D}\nabla.$$

The square-set $u\pm u'$, u' indefinitely small: differential formulæ of the second order.

52. I consider the original form

$$\mathrm{C}^2 0\,\mathrm{C}(u+u')\mathrm{C}(u-u')=\mathrm{C}^2u\mathrm{C}^2u'-\mathrm{D}^2u\mathrm{D}^2u',$$

(which is of course included in the last-mentioned equations).

Writing this in the form

$$\mathrm{C}^2 0\frac{\mathrm{C}(u+u')\mathrm{C}(u-u')}{\mathrm{C}^2u}=\mathrm{C}^2u'-\frac{\mathrm{D}^2u\mathrm{D}^2u'}{\mathrm{C}^2u},$$

and taking u' indefinitely small, whence

$$\begin{aligned}\mathrm{C}(u+u')&=\mathrm{C}u+u'\mathrm{C}'u+\tfrac{1}{2}u'^2\mathrm{C}''u,\ \mathrm{C}u'=\mathrm{C}0,\\ \mathrm{C}(u-u')&=\mathrm{C}u-u'\mathrm{C}'u+\tfrac{1}{2}u'^2\mathrm{C}''u,\ \mathrm{D}u'=\quad u'\mathrm{D}'0,\\ \mathrm{C}(u+u')\mathrm{C}(u-u')&=\mathrm{C}^2u+u'^2\{\mathrm{C}u\mathrm{C}''u-(\mathrm{C}'u)^2\},\end{aligned}$$

the equation becomes

$$\mathrm{C}^2 0\left(1+u'^2\left\{\frac{\mathrm{C}''u}{\mathrm{C}u}-\left(\frac{\mathrm{C}'u}{\mathrm{C}u}\right)^2\right\}\right)=\mathrm{C}^2 0+u'^2\left\{\mathrm{C}0\mathrm{C}''0-(\mathrm{D}'0)^2\frac{\mathrm{D}^2u}{\mathrm{C}^2u}\right\},$$

that is

$$\frac{\mathrm{C}''u}{\mathrm{C}u}-\left(\frac{\mathrm{C}'u}{\mathrm{C}u}\right)^2=\frac{\mathrm{C}''0}{\mathrm{C}0}-\left(\frac{\mathrm{D}'0}{\mathrm{C}0}\right)^2\frac{\mathrm{D}^2u}{\mathrm{C}^2u},$$

viz., we have $\left(\frac{d}{du}\right)^2\log \mathrm{C}u$ expressed in terms of the quotient-function $\frac{\mathrm{D}^2u}{\mathrm{C}^2u}$, and consequently $\mathrm{C}u$ given as an exponential, the argument of which depends on the double integral $\int du\int du\,\frac{\mathrm{D}^2u}{\mathrm{C}^2u}$.

53. To complete the result I write the equation in the form

$$\frac{d^2}{du^2}\log \mathrm{C}u=\frac{\mathrm{C}''0}{\mathrm{C}0}-\frac{1}{k}\left(\frac{\mathrm{D}'0}{\mathrm{C}0}\right)^2+\frac{1}{k}\left(\frac{\mathrm{D}'0}{\mathrm{C}0}\right)^2\left(1-k\,\frac{\mathrm{D}^2u}{\mathrm{C}^2u}\right);$$

$\frac{\mathrm{D}'0}{\mathrm{C}'0}$ is $=-\sqrt{k}\mathrm{K}$, and $\frac{\mathrm{C}''0}{\mathrm{C}0}$ is $=\mathrm{K}(\mathrm{K}-\mathrm{E})$; hence the equation is

$$\frac{d^2}{du^2}\log \mathrm{C}u=\mathrm{K}^2\left(1-\frac{\mathrm{E}}{\mathrm{K}}-k\,\frac{\mathrm{D}^2u}{\mathrm{C}^2u}\right),\ =\mathrm{K}^2\left(1-\frac{\mathrm{E}}{\mathrm{K}}-k^2\mathrm{sn}^2\mathrm{K}u\right),$$

or integrating twice, observing that $\frac{d}{du}\log \mathrm{C}u$ and $\log \mathrm{C}u$, for $u=0$, become $=0$ and $\log \mathrm{C}0$ respectively,

$$\log \mathrm{C}u=\log \mathrm{C}0+\tfrac{1}{2}\left(1-\frac{\mathrm{E}}{\mathrm{K}}\right)\mathrm{K}^2u^2-k^2\int_0 du\int_0 du\,\mathrm{K}^2\mathrm{sn}^2\mathrm{K}u,$$

which is in fact

$$\log \Theta(\mathrm{K}u)=\log \mathrm{C}0+\tfrac{1}{2}\left(1-\frac{\mathrm{E}}{\mathrm{K}}\right)\mathrm{K}^2u^2-k^2\int_0 du\int_0 du\,\mathrm{K}^2\mathrm{sn}^2\mathrm{K}u,$$

agreeing with Jacobi's

$$\log \Theta u=\log \Theta 0+\tfrac{1}{2}\left(1-\frac{\mathrm{E}}{\mathrm{K}}\right)u^2-k^2\int_0 du\int_0 du\,\mathrm{sn}^2u.$$

Elliptic integrals of the third kind.

54. We may write

$$\frac{\mathrm{A}(u+u')\mathrm{A}(u-u')}{\mathrm{A}^2u\mathrm{A}^2u'}=\frac{1}{\mathfrak{A}\mathrm{gh}}\,\frac{\nabla_1}{a-x.a-x'},$$

$$\frac{\mathrm{B}(u+u')\mathrm{B}(u-u')}{\mathrm{B}^2u\mathrm{B}^2u'}=\frac{1}{\mathfrak{B}\mathrm{hf}}\,\frac{\nabla_2}{b-x.b-x'},$$

$$\frac{\mathrm{C}(u+u')\mathrm{C}(u-u')}{\mathrm{C}^2u\mathrm{C}^2u'}=\frac{1}{\mathfrak{C}\mathrm{fg}}\,\frac{\nabla_3}{c-x.c-x'},$$

$$\frac{\mathrm{D}(u+u')\mathrm{D}(u-u')}{\mathrm{D}^2u\mathrm{D}^2u'}=\frac{1}{\mathfrak{D}}\,\frac{x-x'}{d-x.d-x'}.$$

We differentiate logarithmically in regard to u'. Observing that

$$\mathrm{K}du'=\frac{\tfrac{1}{2}\sqrt{\mathrm{af}}dx'}{\sqrt{a-x'.b-x'.c-x'.d-x'}},\ =\frac{\tfrac{1}{2}\sqrt{\mathrm{af}}}{\sqrt{\mathrm{X}'}}dx'$$

suppose, the first equation gives

$$\frac{\mathrm{A}'u}{\mathrm{A}u}+\tfrac{1}{2}\frac{\mathrm{A}'(u-u')}{\mathrm{A}(u+u')}-\tfrac{1}{2}\frac{\mathrm{A}'(u+u')}{\mathrm{A}(u+u')}=-\frac{\mathrm{K}\sqrt{\mathrm{X}'}}{\sqrt{\mathrm{af}}}\frac{d}{dx'}\log\frac{\nabla_1}{a-x'},$$

and if for a moment

$$\nabla_1,=\begin{vmatrix}1, & x+x', & xx'\\ 1, & a+d, & ad\\ 1, & b+c, & bc\end{vmatrix},\ \text{is put }=\mathrm{P}(a-x')+\mathrm{Q}(d-x'),$$

then

$$\frac{d}{dx'}\log\frac{\nabla_1}{a-x'},\ =\frac{d}{dx'}\log\left(\mathrm{P}+\mathrm{Q}\frac{d-x'}{a-x'}\right)\text{ is }=\frac{\mathrm{Q}(d-a)}{(a-x')\nabla_1},\ =-\frac{\mathrm{Qf}}{(a-x')\nabla_1}.$$

But writing $x'=a$ we have

$$\mathrm{Q}(d-a),\ =-\mathrm{Qf}=\begin{vmatrix}1, & a+x, & ax\\ 1, & a+d, & ad\end{vmatrix},\ =(a-b)(a-c)(d-x),\ =-\mathrm{bc}(d-x),$$

that is, $\mathrm{Qf}=-\mathrm{bc}(d-x)$, or

$$\frac{d}{dx'}\log\frac{\nabla_1}{a-x'}=\frac{\mathrm{bc}(d-x')}{(a-x')\nabla_1}.$$

Hence the equation is

$$2\frac{\mathrm{A}'(u')}{\mathrm{A}(u')}+\frac{\mathrm{A}'(u-u')}{\mathrm{A}(u-u')}-\frac{\mathrm{A}'(u+u')}{\mathrm{A}(u+u')}=\frac{2\mathrm{Kbc}}{\sqrt{\mathrm{af}}}\sqrt{\mathrm{X}'}\frac{d-x}{(a-x')\nabla_1};$$

and similarly

$$2\frac{\mathrm{B}'(u')}{\mathrm{B}(u')}+\frac{\mathrm{B}'(u-u')}{\mathrm{B}(u-u')}-\frac{\mathrm{B}'(u+u')}{\mathrm{B}(u+u')}=\frac{2\mathrm{Kca}}{\sqrt{\mathrm{af}}}\sqrt{\mathrm{X}'}\frac{d-x}{(b-x')\nabla_2},$$

$$2\frac{\mathrm{C}'(u')}{\mathrm{C}(u')}+\frac{\mathrm{C}'(u-u')}{\mathrm{C}(u-u')}-\frac{\mathrm{C}'(u+u')}{\mathrm{C}(u+u')}=\frac{2\mathrm{Kab}}{\sqrt{\mathrm{af}}}\sqrt{\mathrm{X}'}\frac{d-x}{(c-x')\nabla_3},$$

$$2\frac{\mathrm{D}'(u')}{\mathrm{D}(u')}+\frac{\mathrm{D}'(u-u')}{\mathrm{D}(u-u')}-\frac{\mathrm{D}'(u+u')}{\mathrm{D}(u+u')}=\frac{2\mathrm{K}}{\sqrt{\mathrm{af}}}\sqrt{\mathrm{X}'}\frac{d-x}{(d-x')(x-x')}.$$

55. Multiply each of these equations by $du,\ =\tfrac{1}{2}\frac{\sqrt{\mathrm{af}}}{\mathrm{K}}\frac{dx}{\sqrt{\mathrm{X}}}$, and integrate. We have equations such as

$$2u\frac{\mathrm{A}'(u')}{\mathrm{A}(u')}+\log\frac{\mathrm{A}(u-u')}{\mathrm{A}(u+u')}=\text{const.}+\frac{\mathrm{bc}\sqrt{\mathrm{X}'}}{\sqrt{\mathrm{af}(a-x')}}\int\frac{(d-x)dx}{\nabla_1\sqrt{\mathrm{X}}},$$

showing how the integrals of the third kind

$$\int\frac{(d-x)dx}{\nabla_1\sqrt{\mathrm{X}}},\quad\int\frac{(d-x)dx}{\nabla_2\sqrt{\mathrm{X}}},\quad\int\frac{(d-x)dx}{\nabla_3\sqrt{\mathrm{X}}},\quad\int\frac{(d-x)dx}{(x-x')\sqrt{\mathrm{X}}}$$

depend on the theta-functions.

If, instead, we work with the original equation

$$\mathrm{C}^2 0\,\frac{\mathrm{C}(u+u')\mathrm{C}(u-u')}{\mathrm{C}^2u.\mathrm{C}^2u'}=1-\frac{\mathrm{D}^2u}{\mathrm{C}^2u}\frac{\mathrm{D}^2u'}{\mathrm{C}^2u'},$$

we find in the same way

$$\begin{aligned}2\frac{\mathrm{C}'(u')}{\mathrm{C}(u')}+\frac{\mathrm{C}'(u-u')}{\mathrm{C}(u-u')}-\frac{\mathrm{C}(u+u')}{\mathrm{C}(u+u')}&=-\frac{d}{du'}\log\left(1-\frac{\mathrm{D}^2u\mathrm{D}^2u'}{\mathrm{C}^2u\mathrm{C}^2u'}\right),\\&=-\frac{d}{du'}\log\left(1-k^2\mathrm{sn}^2\mathrm{K}u\ \mathrm{sn}^2\mathrm{K}u'\right),\\&=\frac{2k^2\mathrm{K}\mathrm{sn}\mathrm{K}u'\mathrm{cn}\mathrm{K}u'\mathrm{dn}\mathrm{K}u'\mathrm{sn}^2\mathrm{K}u}{1-k^2\mathrm{sn}^2\mathrm{K}u'\mathrm{sn}^2\mathrm{K}u};\end{aligned}$$

or multiplying by $\frac{1}{2}du$ and integrating

$$u\frac{\mathrm{C}'(u')}{\mathrm{C}(u')}+\tfrac{1}{2}\log\frac{\mathrm{C}(u-u')}{\mathrm{C}(u+u')}=\int\frac{k^2\mathrm{sn}\mathrm{K}u'\mathrm{cn}\mathrm{K}u'\mathrm{dn}\mathrm{K}u'\mathrm{sn}^2\mathrm{K}u.\mathrm{K}du}{1-k^2\mathrm{sn}^2\mathrm{K}u'\mathrm{sn}^2\mathrm{K}u},$$

which is in fact JACOBI'S equation

$$u\frac{\Theta'a}{\Theta a}+\tfrac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)}=\int\frac{\mathrm{sn}\,a\ \mathrm{cn}\,a\ \mathrm{dn}\,a\ \mathrm{sn}^2u\ du}{1-k^2\mathrm{sn}^2a\ \mathrm{sn}^2u},\ =\Pi(u,\ a).$$

I do not effect the operation but consider the forms first obtained,

$$\mathrm{A}(u+u')\mathrm{A}(u-u')=\frac{\mathfrak{A}}{\mathrm{gh}}\nabla_1,\ \&\text{c.},$$

as the equivalent of JACOBI'S last-mentioned equation.

Addition-formulæ.

56. The addition-theorem for the quotient-functions is of course given by means of the theorem for the elliptic functions: but more elegantly by the formulæ relating to

the form $dx \div \sqrt{a-x.b-x.c-x.d-x}$ obtained in my paper "On the Double ϑ-Functions" ('Crelle-Borchardt,' tom. 87 (1879), pp. 74–81), viz.: for the differential equation

$$\frac{dx}{\sqrt{\mathrm{X}}}+\frac{dy}{\sqrt{\mathrm{Y}}}-\frac{dz}{\sqrt{\mathrm{Z}}}=0,$$

to obtain the particular integral which for $y=d$ reduces itself to $z=x$, we must, in the formulæ of the paper just referred to, interchange a and d: and writing for shortness a, b, c, $\mathrm{d}=a-x$, $b-x$, $c-x$, $d-x$, and similarly $\mathrm{a}_{,}$, $\mathrm{b}_{,}$, $\mathrm{c}_{,}$, $\mathrm{d}_{,}=a-y$, $b-y$, $c-y$, $d-y$, then when the interchange is made, the formulæ become

$$\sqrt{\frac{d-z}{a-z}}$$

$$=\frac{\sqrt{d-b.d-c}\{\sqrt{\mathrm{adb}_{,}\mathrm{c}_{,}}+\sqrt{\mathrm{a}_{,}\mathrm{d}_{,}\mathrm{bc}}\}}{(bc,\ ad)},$$

$$=\frac{\sqrt{d-b.d-c.(x-y)}}{\sqrt{\mathrm{adb}_{,}\mathrm{c}_{,}}-\sqrt{\mathrm{a}_{,}\mathrm{d}_{,}\mathrm{bc}}},$$

$$=\frac{\sqrt{d-b.d-c}\{\sqrt{\mathrm{bdc}_{,}\mathrm{a}_{,}}+\sqrt{\mathrm{b}_{,}\mathrm{d}_{,}\mathrm{ca}}\}}{(d-c)\sqrt{\mathrm{aba}_{,}\mathrm{b}_{,}}-(b-a)\sqrt{\mathrm{cdc}_{,}\mathrm{d}_{,}}},$$

$$=\frac{\sqrt{d-b.d-c}\{\sqrt{\mathrm{cda}_{,}\mathrm{b}_{,}}+\sqrt{\mathrm{abc}_{,}\mathrm{d}_{,}}\}}{(d-b)\sqrt{\mathrm{aca}_{,}\mathrm{c}_{,}}-(c-a)\sqrt{\mathrm{bdb}_{,}\mathrm{d}_{,}}},$$

$$\sqrt{\frac{b-z}{a-z}}$$

$$=\frac{\sqrt{\frac{d-b}{d-a}}\{(d-c)\sqrt{\mathrm{aba}_{,}\mathrm{b}_{,}}+(b-d)\sqrt{\mathrm{cdc}_{,}\mathrm{d}_{,}}\}}{(bc,\ ad)},$$

$$=\frac{\sqrt{\frac{d-b}{d-a}}\{\sqrt{\mathrm{bda}_{,}\mathrm{c}}-\sqrt{\mathrm{b}_{,}\mathrm{d}_{,}\mathrm{ac}}\}}{\sqrt{\mathrm{adb}_{,}\mathrm{c}_{,}}-\sqrt{\mathrm{a}_{,}\mathrm{d}_{,}\mathrm{bc}}},$$

$$=\frac{\sqrt{\frac{d-b}{d-a}}(ac,\ bd)}{(d-c)\sqrt{\mathrm{aba}_{,}\mathrm{b}_{,}}-(b-a)\sqrt{\mathrm{cdc}_{,}\mathrm{d}_{,}}},$$

$$=\frac{\sqrt{\frac{d-b}{d-a}}\{(d-a)\sqrt{\mathrm{bcb}_{,}\mathrm{c}}+(b-c)\sqrt{\mathrm{aba}_{,}\mathrm{b}_{,}}\}}{(d-b)\sqrt{\mathrm{aca}_{,}\mathrm{c}_{,}}-(c-a)\sqrt{\mathrm{bdb}_{,}\mathrm{d}_{,}}},$$

$$\sqrt{\frac{c-z}{a-z}}$$

$$=\frac{\sqrt{\frac{d-c}{d-a}}\{(d-b)\sqrt{\mathrm{cac}_{,}\mathrm{a}_{,}}+(c-a)\sqrt{\mathrm{bdb}_{,}\mathrm{d}_{,}}\}}{(bc,\ ad)},$$

$$=\frac{\sqrt{\frac{d-c}{d-a}}\{\sqrt{\mathrm{cda}_{,}\mathrm{b}_{,}}-\sqrt{\mathrm{abc}_{,}\mathrm{d}_{,}}\}}{\sqrt{\mathrm{adb}_{,}\mathrm{c}_{,}}-\sqrt{\mathrm{a}_{,}\mathrm{d}_{,}\mathrm{bc}}},$$

$$=\frac{\sqrt{\frac{d-c}{d-a}}\{(d-a)\sqrt{\mathrm{bcb}_{,}\mathrm{c}_{,}}-(b-c)\sqrt{\mathrm{ada}_{,}\mathrm{d}_{,}}}{(d-c)\sqrt{\mathrm{aba}_{,}\mathrm{b}_{,}}-(b-a)\sqrt{\mathrm{cdc}_{,}\mathrm{d}_{,}}},$$

$$=\frac{\sqrt{\frac{d-c}{d-a}}(ab,\ cd)}{(d-b)\sqrt{\mathrm{aca}_{,}\mathrm{c}_{,}}-(c-a)\sqrt{\mathrm{bdb}_{,}\mathrm{d}_{,}}}.$$

57. In the foregoing formulæ $(bc,\ ad)$ $(ac,\ bd)$ and $(ad,\ bc)$ denote respectively

$$\begin{vmatrix} 1, & x+y, & xy \\ 1, & b+c, & bc \\ 1, & a+d, & ad \end{vmatrix}\ ,\quad \begin{vmatrix} 1, & x+y, & xy \\ 1, & c+a, & ca \\ 1, & b+d, & bd \end{vmatrix}\ ,\quad \begin{vmatrix} 1, & x+y, & xy \\ 1, & a+b, & ab \\ 1, & c+d, & cd \end{vmatrix}\ ;$$

and substituting for $\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D}$ their values, and for a, b, &c., writing again $a-x$, $b-x$, &c., we have moreover

$$\begin{array}{ll|lll}
\mathrm{A}^2u=\sqrt{c-b.b-d.c-d} & (a-x), & \mathrm{A}^2v=\sqrt{\qquad ,, \qquad} & & (a-y), \\
\mathrm{B}^2u=\sqrt{c-a.c-d.a-d} & (b-x), & \mathrm{B}^2v=\sqrt{\qquad ,, \qquad} & & (b-y), \\
\mathrm{C}^2u=\sqrt{a-b.a-d.b-d} & (c-x), & \mathrm{C}^2v=\sqrt{\qquad ,, \qquad} & & (c-y), \\
\mathrm{D}^2u=\sqrt{c-b.c-a.a-b} & (d-x), & \mathrm{D}^2v=\sqrt{\qquad ,, \qquad} & & (d-y),
\end{array}$$

$$\begin{array}{ll}
\mathrm{A}^2(u+v)=\sqrt{\qquad ,, \qquad} & (a-z), \\
\mathrm{B}^2(u+v)=\sqrt{\qquad ,, \qquad} & (b-z), \\
\mathrm{C}^2(u+v)=\sqrt{\qquad ,, \qquad} & (c-z), \\
\mathrm{D}^2(u+v)=\sqrt{\qquad ,, \qquad} & (d-z),
\end{array}$$

the constant multipliers being of course the same in the three columns respectively. According to what precedes, the radical of the fourth line should be taken with the sign $-$. The functions $(bc,\ ad)$, &c., contained in the formulæ require a transformation such as

$$(b-c)\ (bc,\ ad)=\begin{vmatrix} b-x.b-y, & c-x.c-y \\ b-a.b-d, & c-a.c-d \end{vmatrix}$$

in order to make them separately homogeneous in the differences $a-x$, $b-x$, $c-x$, $d-x$, and $a-y$, $b-y$, $c-y$, $d-y$, and therefore expressible as linear functions of the squared functions A^2u, &c., and A^2v, &c., respectively: and the formulæ then give the quotient-functions $\mathrm{A}(u+v)\div\mathrm{D}(u+v)$ &c., in terms of the quotient-functions of u and v respectively.

Doubly infinite product-forms.

58. The functions $\mathrm{A}u$, $\mathrm{B}u$, $\mathrm{C}u$, $\mathrm{D}u$ may be expressed each as a doubly infinite product. Writing for shortness

$$m \quad + \quad n.\frac{a}{\pi i}=(m,\ n),$$

$$m+1+ \quad n.\frac{a}{\pi i}=(\bar{m},\ n),$$

$$m \quad +(n+1)\frac{a}{\pi i}=(m,\ \bar{n}),$$

$$m+1+(n+1)\frac{a}{\pi i}=(\bar{m},\ \bar{n}),$$

then the formulæ are

$$\mathrm{A}u=\mathrm{A}0.\ \Pi\Pi\left\{1+\frac{u}{(\bar{m},\ \bar{n})}\right\},$$

$$\mathrm{B}u=\mathrm{B}0.\ \Pi\Pi\left\{1+\frac{u}{(\bar{m},\ n)}\right\},$$

$$\mathrm{C}u=\mathrm{C}0.\ \Pi\Pi\left\{1+\frac{u}{(m,\ \bar{n})}\right\},$$

$$\mathrm{D}u=\mathrm{D}'0.u\Pi\Pi\left\{1+\frac{u}{(m,\ n)}\right\},$$

where in all the formulæ m, n denote even integers having all values whatever, zero included, from $-\infty$ to $+\infty$; only in the formula for Du, the term for which m and n are simultaneously $=0$, is to be omitted.

59. But a further definition in regard to the limits is required: first, we assume that m has the corresponding positive and negative values, and similarly that n has corresponding positive and negative values*; or say, in the four formulæ respectively, we consider m, n as extending

m from $-\mu$ to $\mu+2$, n from $-\nu$ to $\nu+2$,
,, ,, $-\mu$,, $\mu+2$, ,, ,, $-\nu$,, ν,
,, ,, $-\mu$,, μ , ,, ,, $-\nu$,, $\nu+2$,
,, ,, $-\mu$,, μ , ,, ,, $-\nu$,, ν,

where μ and ν are each of them ultimately infinite. But, secondly, it is necessary that μ should be indefinitely larger than ν, or say that ultimately $\frac{\nu}{\mu}=0$.

60. In fact, transforming the q-series into products as in the 'Fundamenta Nova,' and neglecting for the moment mere constant factors, we have

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[*] This is more than is necessary; it would be enough if the ultimate values of m were $-\mu$, μ', μ and μ' being in a ratio of equality; and the like as regards n. But it is convenient that the numbers should be absolutely equal.

$$\begin{aligned}
\mathrm{A}u &= \qquad (1+2q\cos\pi u+q^2)(1+2q^3\cos\pi u+q^6)\ldots,\\
\mathrm{B}u &= \cos\tfrac{1}{2}\pi u(1+2q^2\cos\pi u+q^4)(1+2q^4\cos\pi u+q^8)\ldots,\\
\mathrm{C}u &= \qquad (1-2q\cos\pi u+q^2)(1-2q^3\cos\pi u+q^6),\\
\mathrm{D}u &= \sin\tfrac{1}{2}\pi u(1-2q^2\cos\pi u+q^4)(1-2q^4\cos\pi u+q^8),
\end{aligned}$$

and writing for a moment $\alpha=\frac{a}{\pi i}$ and therefore $q^{\frac{1}{2}}+q^{-\frac{1}{2}},=e^{\frac{1}{2}\pi\alpha i}+e^{-\frac{1}{2}\pi\alpha i},=2\cos\frac{1}{2}\pi\alpha$, &c., each of these expressions is readily converted into a singly infinite product of sines or cosines

$$\begin{aligned}
\mathrm{A}u &= \Pi.\cos\tfrac{1}{2}\pi(u+\overline{n}\alpha),\\
\mathrm{B}u &= \Pi.\cos\tfrac{1}{2}\pi(u+n\alpha),\\
\mathrm{C}u &= \Pi.\sin\tfrac{1}{2}\pi(u+\overline{n}\alpha),\\
\mathrm{D}u &= \Pi.\sin\tfrac{1}{2}\pi(u+n\alpha),
\end{aligned}$$

where $\overline{n}$ is written to denote $n+1$, and n has all positive or negative even values (zero included) from $-\infty$ to $+\infty$, or more accurately from $-\nu$ to ν, if ν is ultimately infinite.

61. The sines and cosines are converted into infinite products by the ordinary formulæ, which neglecting constant factors may conveniently be written

$$\sin\tfrac{1}{2}\pi u=\Pi(u+m),\ \cos\tfrac{1}{2}\pi u=\Pi(u+\overline{m}),$$

where $\overline{m}$ is written to denote $m+1$, and m has all positive or negative even values (zero included) from $-\infty$ to $+\infty$, or more accurately from $-\mu$ to μ, if μ be ultimately infinite. But in applying these formulæ to the case of a function such as $\sin\frac{1}{2}\pi(u+n\alpha)$, it is assumed that the limiting values $\mu, -\mu$ of m are indefinitely large in regard to $u+n\alpha$; and therefore, since n may approach to its limiting value $\pm\nu$, it is necessary that μ should be indefinitely large in comparison with ν, or that $\frac{\nu}{\mu}=0$.

62. It is on account of this unsymmetry as to the limits $\frac{\nu}{\mu}=0, \frac{\mu}{\nu}=\infty$, that we have 1 as a quarter period, $\frac{a}{\pi i}$ only as a quarter-quasi-period of the single theta-functions.

The transformation q to r, $\log q \log r=\pi^2$.

63. In general, if we consider the ratio of two such infinite products where for the first the limits are $(\pm\mu, \pm\nu)$, and for the second they are $(\pm\mu', \pm\nu')$, and where for convenience we take $\mu>\mu'$, $\nu>\nu'$, then the quotient, say $[\mu, \nu]\div[\mu', \nu]$ is $=$ exp. $(\mathrm{M}u^2)$ where

$$\mathrm{M}=-\tfrac{1}{8}\iint\frac{dm\,dn}{(m,n)^2}$$

taken over the area included between the two rectangles. We have $(m, n)=m+\frac{a}{\pi i}n$, $=m+i\theta n$ suppose, where (a being negative) $\theta=-\frac{a}{\pi}$, is positive: the integral is

$$\iint\frac{dm\,dn}{(m+i\theta n)^2},=\int dm.-\frac{1}{i\theta}\left(\frac{1}{m+i\theta n}\right)_{-\nu}^{\nu},$$

$$=\frac{1}{i\theta}\int dm\left(\frac{1}{m-i\theta\nu}-\frac{1}{m+i\theta\nu}\right),$$

$$=\frac{1}{i\theta}\log\frac{m-i\theta\nu}{m+i\theta\nu};$$

or finally between the proper limits the value is

$$=\frac{2}{i\theta}\left\{\log\left(\frac{\mu-i\theta\nu}{\mu+i\theta\nu}\right)-\log\left(\frac{\mu'-i\theta\nu'}{\mu'+i\theta\nu'}\right)\right\},$$

where the logarithms are $\log(\mu-i\theta\nu)=\log\sqrt{\mu^2+\nu^2}-i\tan^{-1}\frac{\theta\nu}{\mu}$, &c., and the $\tan^{-1}$ denotes always an arc between the limits $-\frac{1}{2}\pi$, $+\frac{1}{2}\pi$. Hence if $\frac{\mu}{\nu}=\infty$, $\frac{\mu'}{\nu'}=0$, the value is $\frac{2}{i\theta}(-0i-0i+\frac{1}{2}\pi i+\frac{1}{2}\pi i)=\frac{2\pi}{\theta},=-\frac{2\pi^2}{a}$; or $\mathrm{K}=\frac{1}{4}\frac{\pi^2}{a}$. Hence finally

$$[\mu\div\nu,=\infty]\div[\mu\div\nu,=0]=\exp\left(\tfrac{1}{4}\frac{\pi^2}{a}u^2\right).$$

64. We have $a,=\log q$, negative; hence taking r such that $\log q\log r=\pi^2$, we have $a'=\log r$, also negative; and r, like q, is positive and less than 1. We consider the theta-functions which depend on r in the same manner that the original functions did on q, say these are

$$\mathrm{A}(u,\ r)=\mathrm{A}\ (0,\ r)\ \Pi\Pi\left\{1+\frac{u}{\overline{m}+\overline{n}\frac{a'}{\pi i}}\right\},$$

$$\mathrm{B}(u,\ r)=\mathrm{B}\ (0,\ r)\ \Pi\Pi\left\{1+\frac{u}{\overline{m}+n\frac{a'}{\pi i}}\right\},$$

$$\mathrm{C}(u,\ r)=\mathrm{C}\ (0,\ r)\ \Pi\Pi\left\{1+\frac{u}{m+\overline{n}\frac{a'}{\pi i}}\right\},$$

$$\mathrm{D}(u,\ r)=\mathrm{D}'(0,\ r)u\Pi\Pi\left\{1+\frac{u}{m+n\frac{a'}{\pi i}}\right\},$$

limits as before, and in particular $\frac{\mu}{\nu}=\infty$; it is at once seen that if in the original functions, which I now call $A(u, q)$, $B(u, q)$, $C(u, q)$, $D(u, q)$, we write $\frac{au}{\pi i}$ for u, we obtain the same infinite products which present themselves in the expressions of the new functions $A(u, r)$, &c., only with a different condition as to the limits; we have for instance

$$\Pi\Pi\left(1+\frac{\frac{au}{\pi i}}{m+n\frac{a}{\pi i}}\right)=\Pi\Pi\left(1+\frac{u}{n-m\frac{a'}{\pi i}}\right),=\Pi\Pi\left(1+\frac{u}{n+m\frac{a'}{\pi i}}\right),$$

which, interchanging m, n, and of course also μ, ν, is

$$=\Pi\Pi\left(1+\frac{u}{m+n\frac{a'}{\pi i}}\right),$$

with the condition $\frac{\mu}{\nu}=0$ instead of $\frac{\mu}{\nu}=\infty$. Hence disregarding for the moment constant factors, and observing that a is replaced by a', we have

$$D(u, r)\div D\left(\frac{au}{\pi i}, q\right)=[\mu\div\nu,=\infty]\div[\mu\div\nu,=0]$$

$$=\exp\left(\tfrac{1}{4}\frac{\pi^2}{a'}u^2\right),=\exp\left(\tfrac{1}{4}u^2\log q\right).$$

65. We have equations of this form for the four functions, but with a proper constant multiplier in each equation: the equations, in fact, are

$$A(u, r)=\{A(0, r)\div A(0, q)\}\exp\left(\tfrac{1}{4}u^2\log q\right)A\left(\frac{au}{\pi i}, q\right),$$

$$B(u, r)=\{B(0, r)\div B(0, q)\}\quad ,, \quad B\left(\frac{au}{\pi i}, q\right),$$

$$C(u, r)=\{C(0, r)\div C(0, q)\}\quad ,, \quad C\left(\frac{au}{\pi i}, q\right),$$

$$D(u, r)=\{D'(0, r)\div D'(0, q)\}\frac{\pi i}{a}\quad ,, \quad D\left(\frac{au}{\pi i}, q\right).$$

It is to be observed that r is the same function of k' that q is of k: this would at once follow from Jacobi's equation $\log q=-\frac{\pi K'}{K}$, for then $\log q\log r=\pi^2$ and therefore $\log r=-\frac{\pi K'}{K}$ (only we are not at liberty to use the relation in question $\log q=-\frac{\pi K'}{K}$), and assuming it to be true we have

$$k=\frac{B^2(0,q)}{A^2(0,q)},\ k'=\frac{C^2(0,q)}{A^2(0,q)},\ K=-\frac{A(0,q)D'(0,q)}{B(0,q)C(0,q)},$$

$$k=\frac{C^2(0,r)}{A^2(0,r)},\ k'=\frac{B^2(0,r)}{A^2(0,r)},\ K'=-\frac{A(0,r)D'(0,r)}{B(0,r)C(0,r)},$$

$$\log q=-\frac{\pi K'}{K},\ \log r=-\frac{\pi K}{K'},$$

where if the identity of the two values of k or of the two values of k' were proved independently (as might doubtless be done), the required theorem (r the same function of k' that q is of k) would follow conversely: and thence the other equations of the system.

66. We have in the 'Fundamenta Nova,' p. 75, the equation

$$\frac{H(iu,k)}{\Theta(0,k)}=i\sqrt{\frac{k}{k'}}e^{\frac{\pi u^2}{4KK'}}\frac{H(u,k')}{\Theta(0,k')};$$

writing here $K'u$ instead of u the equation becomes

$$\frac{H(iK'u,k)}{\Theta(0,k)}=i\sqrt{\frac{k}{k'}}\exp\left(\tfrac{1}{4}\frac{\pi K'}{K}u^2\right).\frac{H(K'u,k')}{\Theta(0,k')}$$

or what is the same thing

$$\frac{D\left(\frac{au}{\pi i},q\right)}{C(0,q)}=i\sqrt{\frac{k}{k'}}\exp\left(-\tfrac{1}{4}u^2\log q\right).\frac{D(u,r)}{C(0,r)}$$

which can be readily identified with the foregoing equation between $D\left(\frac{au}{\pi i},q\right)$ and $D(u,r)$. But the real meaning of the transformation is best seen by means of the double-product formulæ.

THIRD PART.—THE DOUBLE THETA-FUNCTIONS.

Notations, &c.

67. We have here 16 functions $\vartheta\begin{pmatrix}\alpha\beta\\ \gamma\delta\end{pmatrix}(u,v)$: this notation by characteristics, containing each of them four numbers, is too cumbrous for ordinary use, and I therefore replace it by the current-number notation, in which the characteristics are denoted by the series of numbers 0, 1, 2, . . . 15 : we cannot in place of this introduce the single-and-double-letter notation A, B, . . . AB, &c., for there is not here any correspondence of the two notations, nor is there anything in the definition of the functions which in

anywise suggests the single-and-double-letter notation: this first presents itself in connexion with the relations between the functions given by the product-theorem: and as the product theorem is based upon the notation by characteristics, it is proper to present the theorem in this notation, or in the equivalent current-number notation: and then to show how by the relations thus obtained between the functions we are led to the single-and-double-letter notation.

68. There are some other notations which may be referred to: and for showing the correspondence between them I annex the following table:—

THE double theta-functions.

Asterisk denotes the odd functions.	1. Current number.	2. Character.	3. Single-and-double-letter, CAYLEY.	4. GÖPEL.	5. GÖPEL-CAYLEY.	6. ROSENHAIN.	7. WEIERSTRASS.	8. KUMMER.
	ϑ_0	ϑ^{00}_{00}	BD	P‴	P_3	ϑ_{22}	ϑ_5	12
	1	10 00	CE	R‴	R_3	32	4	8
	2	01 00	CD	Q‴	Q_3	23	01	10
	3	11 00	BE	S‴	S_3	33	23	6
	4	00 10	AC	P′	P_1	02	34	4
*	5	10 10	C	*i*R′	R_1	12	3	16
	6	01 10	AB	Q′	Q_1	03	2	2
*	7	11 10	B	*i*S′	S_1	13	24	14
	8	00 01	BC	P″	P_2	20	12	9
	9	10 01	DE	R″	R_2	30	03	5
*	10	01 01	F	*i*Q″	Q_2	21	02	11
*	11	11 01	A	*i*S″	S_2	31	13	7
	12	00 11	AD	P	P	00	0	1
*	13	10 11	D	*i*R	R	10	04	13
*	14	01 11	E	*i*Q	Q	01	1	3
	15	11 11	AE	S	S	11	14	15

69. These are the notations :—

1. By current-numbers. It may be remarked that the series was taken 0, 1, . . . 15 instead of 1, 2, . . . 16, in order that 0 might correspond to the characteristic $\begin{smallmatrix}00\\00\end{smallmatrix}$;

2. By characteristics;

3. By single-and-double letters ;

4. Göpel's, in his paper above referred to, and

5. The same as used in my paper above referred to ;

6. Rosenhain's, in his paper above referred to ;

7. Weierstrass', as quoted by Königsberger in his paper "Ueber die Transformation der *Abel*schen Functionen erster Ordnung," 'Crelle-Borchardt,' t. 64 (1865), p. 17, and by Borchardt in his paper above referred to ;

8. Not a theta-notation, but the series of current numbers used in Kummer's Memoir "Ueber die algebraischen Strahlen-systeme," 'Berl. Abh.' 1866, for the nodes of his 16-nodal quartic surface, and connected with the double theta-functions in my paper above referred to.

But in the present memoir only the first three columns of the table need be attended to.

70. It will be convenient to introduce here some other remarks as to notation, &c.

The letter c is used in connexion with the zero values $u=0$, $v=0$ of the arguments, viz. :—

$$\vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_6, \vartheta_8, \vartheta_9, \vartheta_{12}, \vartheta_{15}$$

are even functions, and the corresponding zero-functions are denoted by

$$c_0, c_1, c_2, c_3, c_4, c_6, c_8, c_9, c_{12}, c_{15} ;$$

there are thus 10 constants c.

When (u, v) are indefinitely small each of these functions is of course equal to its zero-value *plus* a quadric term in (u, v), and we may write in general

$$\vartheta=c+\tfrac{1}{2}(c''', c^{\text{iv}}, c^{\text{v}}\between u, v)^2 ;$$

there are thus 30 constants c''', c^{iv}, c^{v}.

The remaining functions

$$\vartheta_5, \vartheta_7, \vartheta_{10}, \vartheta_{11}, \vartheta_{13}, \vartheta_{14}$$

are odd functions vanishing for $u=0$, $v=0$, and when these arguments are indefinitely small we may write in general

$$\vartheta=(c', c''\between u, v)$$

there are thus 12 constants c', c''.

71. All these constants are in the first instance given as transcendental functions of the parameters, or say rather of exp. a, exp. h, exp. b (which exponentials correspond to the q of the single theory): viz., in a notation which will be readily understood, the constants c, c''', c^{iv}, c^{v} of the even functions are

$$\Sigma \exp\begin{pmatrix} m+\alpha, & n+\beta \\ \gamma & \delta \end{pmatrix};$$

$$-\tfrac{1}{2}\pi^2\Sigma\ (m+\alpha)^2,\ 2(m+\alpha)(n+\beta),\ (n+\beta)^2,\ \exp\begin{pmatrix} m+\alpha, & n+\delta \\ \gamma & \delta \end{pmatrix};$$

and the constants c', c'' of the odd functions are

$$\tfrac{1}{2}\pi i\Sigma\ (m+\alpha),\ (n+\beta),\ \exp\begin{pmatrix} m+\alpha, & n+\beta \\ \gamma & \delta \end{pmatrix}.$$

72. It is convenient for the verification of results and otherwise, to have the values of the functions, belonging to small values of exp $(-a)$, exp $(-b)$; if to avoid fractional exponents we regard these and exp $(-h)$ as fourth powers, and write

$$\exp(-a)=\mathrm{Q}^4,\ \exp(-h)=\mathrm{R}^4,\ \exp(-b)=\mathrm{S}^4,$$

also

$$\mathrm{QR}^2\mathrm{S}=\Lambda,\ \mathrm{QR}^{-2}\mathrm{S}=\Lambda',\ \text{and therefore } \Lambda\Lambda'=\mathrm{Q}^2\mathrm{S}^2,$$

then attending only to the lowest powers of Q, S we find without difficulty

$$\begin{array}{lll}
\vartheta_0(u)= 1, & \text{and therefore also} & c_0=1, \\
\vartheta_1 = 2\mathrm{Q}\cos\tfrac{1}{2}\pi u, & & c_1=2\mathrm{Q}, \\
\vartheta_2 = 2\mathrm{S}\cos\tfrac{1}{2}\pi v, & & c_2=2\mathrm{S}, \\
\vartheta_3 = 2\Lambda\cos\tfrac{1}{2}\pi(u+v)+2\Lambda'\cos\tfrac{1}{2}\pi(u-v), & & c_3=2\Lambda+2\Lambda', \\
\vartheta_4 = 1, & & c_4=1, \\
\vartheta_5 = -2\mathrm{Q}\sin\tfrac{1}{2}\pi u, & & \\
\vartheta_6 = 2\mathrm{S}\cos\tfrac{1}{2}\pi v, & & c_6=2\mathrm{S}, \\
\vartheta_7 = -2\Lambda\sin\tfrac{1}{2}\pi(u+v)-2\Lambda'\sin\tfrac{1}{2}\pi(u-v), & & \\
\vartheta_8 = 1, & & c_8=1, \\
\vartheta_9 = 2\mathrm{Q}\cos\tfrac{1}{2}\pi u, & & c_9=2\mathrm{Q}, \\
\vartheta_{10} = -2\mathrm{S}\sin\tfrac{1}{2}\pi v, & & \\
\vartheta_{11} = -2\Lambda\sin\tfrac{1}{2}\pi(u+v)+2\Lambda'\sin\tfrac{1}{2}\pi(u-v), & & \\
\vartheta_{12} = 1, & & c_{12}=1, \\
\vartheta_{13} = -2\mathrm{Q}\sin\tfrac{1}{2}\pi u, & & \\
\vartheta_{14} = -2\mathrm{S}\sin\tfrac{1}{2}\pi v, & & \\
\vartheta_{15} = -2\Lambda\cos\tfrac{1}{2}\pi(u+v)+2\Lambda'\cos\tfrac{1}{2}\pi(u-v), & & c_{15}=-2\Lambda+2\Lambda'.
\end{array}$$

73. The expansions might be carried further; we have for instance

$$
\begin{array}{llr}
\vartheta_0(u) = 1+2\mathrm{Q}^4 \cos \pi u + 2\mathrm{S}^4 \cos \pi v, & & c_0 = 1+2\mathrm{Q}^4+2\mathrm{S}^4, \\
\vartheta_4 \quad = 1-2\mathrm{Q}^4 \quad ,, \quad +2\mathrm{S}^4 \quad ,, \quad , & & c_4 = 1-2\mathrm{Q}^4+2\mathrm{S}^4, \\
\vartheta_8 \quad = 1+2\mathrm{Q}^4 \quad ,, \quad -2\mathrm{S}^4 \quad ,, \quad , & & c_8 = 1+2\mathrm{Q}^4-2\mathrm{S}^4, \\
\vartheta_{12} \quad = 1-2\mathrm{Q}^4 \quad ,, \quad -2\mathrm{S}^4 \quad ,, \quad , & & c_{12} = 1-2\mathrm{Q}^4-2\mathrm{S}^4, \\
\vartheta_1 \quad = \quad 2\mathrm{Q} \cos \tfrac{1}{2}\pi u + 2\mathrm{Q}^9 \cos \tfrac{3}{2}\pi u + 2\mathrm{A} \cos \tfrac{1}{2}\pi(u+2v) + 2\mathrm{A}' \cos \tfrac{1}{2}\pi(u-2v), & & \\
& & c_1 = 2\mathrm{Q}+2\mathrm{Q}^9+2\mathrm{A}+2\mathrm{A}', \\
\vartheta_5 \quad = -2\mathrm{Q} \sin \tfrac{1}{2}\pi u + 2\mathrm{Q}^9 \sin \tfrac{3}{2}\pi u - 2\mathrm{A} \sin \tfrac{1}{2}\pi(u+2v) - 2\mathrm{A}' \sin \tfrac{1}{2}\pi(u-2v) & & \\
\vartheta_9 \quad = \quad 2\mathrm{Q} \cos \tfrac{1}{2}\pi u + 2\mathrm{Q}^9 \cos \tfrac{3}{2}\pi u - 2\mathrm{A} \cos \tfrac{1}{2}\pi(u+2v) - 2\mathrm{A}' \cos \tfrac{1}{2}\pi(u-2v), & & \\
& & c_9 = 2\mathrm{Q}+2\mathrm{Q}^9-2\mathrm{A}-2\mathrm{A}', \\
\vartheta_{13} \quad = -2\mathrm{Q} \sin \tfrac{1}{2}\pi u + 2\mathrm{Q}^9 \sin \tfrac{3}{2}\pi u + 2\mathrm{A} \sin \tfrac{1}{2}\pi(u+2v) + 2\mathrm{A}' \sin \tfrac{1}{2}\pi(u-2v), & &
\end{array}
$$

in which last formulæ

$$\mathrm{A} = \mathrm{QR}^4\mathrm{S}^4, \ = \frac{\Lambda^2\mathrm{S}^2}{\mathrm{Q}}; \ \mathrm{A}' = \mathrm{QR}^{-4}\mathrm{S}^4, \ = \frac{\Lambda^{12}\mathrm{S}^2}{\mathrm{Q}}.$$

74. In the single-and-double-letter notation we have six letters A, B, C, D, E, F; and the duads AB, AC, ... DE are used as abbreviations for the double triads ABF, CDE, &c., the letter F always accompanying the expressed duad; there are thus in all six single-letter symbols, and 10 double-letter symbols, in all 16 symbols, corresponding to the double-theta functions, as already mentioned in the order

$$
\begin{array}{lcccccccccccccccc}
\vartheta & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
 & \mathrm{BD}, & \mathrm{CE}, & \mathrm{CD}, & \mathrm{BE}, & \mathrm{AC}, & \mathrm{C}, & \mathrm{AB}, & \mathrm{B}, & \mathrm{BC}, & \mathrm{DE}, & \mathrm{F}, & \mathrm{A}, & \mathrm{AD}, & \mathrm{D}, & \mathrm{E}, & \mathrm{AE}
\end{array}
$$

where observe that the single letters C, B, F, A, D, E correspond to the odd functions 5, 7, 10, 11, 13, 14 respectively.

75. The ground of the notation is as follows:—

The algebraical relations between the double theta-functions are such that introducing six constant quantities a, b, c, d, e, f and two variable quantities (x, y) it is allowable to express the 16 functions as proportional to given functions of these quantities $(a, b, c, d, e, f; x, y)$; viz.: there are six functions the notations of which depend on the single letters a, b, c, d, e, f respectively, and 10 functions the notations of which depend on the pairs ab, ac, ad, ae, bc, bd, be, cd, ce, de respectively: each of the 16 functions is in fact proportional to the product of two factors, viz.: a constant factor depending only on (a, b, c, d, e, f), and a variable factor containing also x and y. Attending in the first instance to the variable factors, and writing for shortness

$$a-x,\ b-x,\ c-x,\ d-x,\ e-x,\ f-x = \mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{d},\ \mathrm{e},\ \mathrm{f} \quad ;\ x-y=\theta;$$

$$a-y,\ b-y,\ c-y,\ d-y,\ e-y,\ f-y = \mathrm{a}_{,},\ \mathrm{b}_{,},\ \mathrm{c}_{,},\ \mathrm{d}_{,},\ \mathrm{e}_{,},\ \mathrm{f}_{,};$$

these are of the forms

$$\sqrt{a}=\sqrt{\mathrm{aa}_{,}},\ \sqrt{ab}=\frac{1}{\theta}\{\sqrt{\mathrm{abfc}_{,}\mathrm{d}_{,}\mathrm{e}_{,}}+\sqrt{\mathrm{a}_{,}\mathrm{b}_{,}\mathrm{f}_{,}\mathrm{cde}}\}$$

and I remark that to avoid ambiguity the squares of these expressions are throughout written as $(\sqrt{a})^2$ and $(\sqrt{ab})^2$ respectively.

76. There is for the constant factors a like single-and-double-letter notation which will be mentioned presently, but in the first instance I use for the even functions the before mentioned 10 letters c, and for the odd ones six letters k. I assume that the values $x, y=\infty, \infty$ (ratio not determined) correspond to the values $u=0, v=0$ of the arguments; and that ω is a function of (x, y) which, when (x, y) are interchanged, changes only its sign, and which for indefinitely large values of (x, y) becomes $=\frac{x-y}{(xy)^{\frac{3}{2}}}$. This being so, we write (adding a second column which will be afterwards explained)

$$\begin{array}{ll}
\vartheta 0=\mathrm{BD}=\omega c_0\sqrt{bd}, & c_0=\lambda\sqrt[4]{\overline{\overline{bd}}}, \\
1=\mathrm{CE}=\text{,,}c_1\sqrt{ce}, & c_1=\text{,,}\sqrt[4]{\overline{\overline{ce}}}, \\
2=\mathrm{CD}=\text{,,}c_2\sqrt{cd}, & c_2=\text{,,}\sqrt[4]{\overline{\overline{cd}}}, \\
3=\mathrm{BE}=\text{,,}c_3\sqrt{be}, & c_3=\text{,,}\sqrt[4]{\overline{\overline{be}}}, \\
4=\mathrm{AC}=\text{,,}c_4\sqrt{ac}, & c_4=\text{,,}\sqrt[4]{\overline{\overline{ac}}}, \\
5=\ \mathrm{C}\ =\text{,,}k_5\sqrt{c}, & k_5=\text{,,}\sqrt[4]{\overline{\overline{c}}}, \\
6=\mathrm{AB}=\text{,,}c_6\sqrt{ab}, & c_6=\text{,,}\sqrt[4]{\overline{\overline{ab}}}, \\
7=\ \mathrm{B}\ =\text{,,}k_7\sqrt{b}, & k_7=\text{,,}\sqrt[4]{\overline{\overline{b}}}, \\
8=\mathrm{BC}=\text{,,}c_8\sqrt{bc}, & c_8=\text{,,}\sqrt[4]{\overline{\overline{bc}}}, \\
9=\mathrm{DE}=\text{,,}c_9\sqrt{de}, & c_9=\text{,,}\sqrt[4]{\overline{\overline{de}}}, \\
10=\ \mathrm{F}\ =\text{,,}k_{10}\sqrt{f}, & k_{10}=\text{,,}\sqrt[4]{\overline{\overline{f}}}, \\
11=\ \mathrm{A}\ =\text{,,}k_{11}\sqrt{a}, & k_{11}=\text{,,}\sqrt[4]{\overline{\overline{a}}}, \\
12=\mathrm{AD}=\text{,,}c_{12}\sqrt{ad}, & c_{12}=\text{,,}\sqrt[4]{\overline{\overline{ad}}}, \\
13=\ \mathrm{D}\ =\text{,,}k_{13}\sqrt{d}, & k_{13}=\text{,,}\sqrt[4]{\overline{\overline{d}}}, \\
14=\ \mathrm{E}\ =\text{,,}k_{14}\sqrt{e}, & k_{14}=\text{,,}\sqrt[4]{\overline{\overline{e}}}, \\
15=\mathrm{AE}=\text{,,}c_{15}\sqrt{ae}, & c_{15}=\text{,,}\sqrt[4]{\overline{\overline{ae}}},
\end{array}$$

viz.: here, on writing $x, y=\infty, \infty$, each of the functions $\sqrt{bd}$, &c. becomes $=2\frac{(xy)^{\frac{3}{2}}}{x-y}$; and each of the functions $\sqrt{a}$, &c., becomes $=\sqrt{xy}$; hence by reason of the assumed form of ω, the odd functions each vanish (their evanescent values being proportional to k_5, k_7, k_{10}, k_{11}, k_{13}, k_{14} respectively), while the even functions become equal to c_0, c_1, c_2, c_3, c_4, c_6, c_8, c_9, c_{12}, c_{15} respectively.

Observe further that on interchanging x, y, the even functions remain unaltered, while the odd functions change their sign; that is, the interchange of x, y corresponds to the change u, v into $-u$, $-v$.

77. As to the values of the 10 c's and the six k's in terms of (a, b, c, d, e, f) these are proportional to fourth roots, $\sqrt{\overline{a}}$, &c., $\sqrt[4]{\overline{ab}}$, &c.; in $\sqrt[4]{\overline{a}}$, a is in the first instance regarded as standing for the pentad $bcdef$, and then this is used to denote a product of differences $bc.bd.be.bf.cd.ce.cf.de.df.ef$; similarly ab is in the first instance regarded as standing for the double triad $abf.cde$, and then each of these triads is used to denote a product of differences, $ab.af.bf$ and $cd.ce.de$ respectively. The order of the letters is always the alphabetical one, viz.: the single letters and duads denote pentads and double triads, thus:

$$\begin{array}{ll}
a=bcdef, & ab=abf.cde, \\
b=acdef, & ac=acf.bde, \\
c=abdef, & ad=adf.bce, \\
d=abcef, & ae=aef.bcd, \\
e=abcdf, & bc=bcf.ade, \\
f=abcde, & bd=bdf.ace, \\
 & be=bef.acd, \\
 & cd=cdf.abe, \\
 & ce=cef.abd, \\
 & de=def.abc.
\end{array}$$

There is no fear of ambiguity in writing (and we accordingly write) the squares of $\sqrt[4]{\overline{a}}$ and $\sqrt[4]{\overline{ab}}$ as $\sqrt{\overline{a}}$ and $\sqrt{\overline{ab}}$ respectively; the fourth powers are written $(\sqrt{\overline{a}})^2$ and $(\sqrt{\overline{ab}})^2$; the double stroke of the radical symbol $\sqrt{\overline{}}$ is in every case perfectly distinctive.

This being so we have as above $c_0=\lambda\sqrt[4]{\overline{bd}}$, &c., $k_5=\lambda\sqrt[4]{\overline{a}}$, &c.: it is, however, important to notice that the fourth roots in question do not denote positive values, but they are fourth roots each taken with its proper sign ($+$, $-$, $+i$, $-i$, as the case may be) so as to satisfy the identical relations which exist between the sixteen constants; and it is not easy to determine the signs.

The x, y are connected with the u, v by the differential relations

$$\sigma du+\tau dv=-\tfrac{1}{2}\left\{\frac{dx}{\sqrt{\mathrm{X}}}-\frac{dy}{\sqrt{\mathrm{Y}}}\right\},$$

$$\varpi du+\rho dv=-\tfrac{1}{2}\left\{\frac{xdx}{\sqrt{\mathrm{X}}}-\frac{ydy}{\sqrt{\mathrm{Y}}}\right\},$$

where X=abcdef, Y=$a_{\prime}b_{\prime}c_{\prime}d_{\prime}e_{\prime}f_{\prime}$; which equations contain the constants ϖ, ρ, σ, τ, the values of which will be afterwards connected with the other constants.

78. The c's are expressed as functions of four quantities α, β, γ, δ, and connected with each other, and with the constants (a, b, c, d, e, f) by the formulæ

$$\begin{array}{lcl}
c^2 & & \\
\overline{0}=\alpha^2+\beta^2+\gamma^2+\delta^2 & = & \omega_0^2\sqrt{bd}, \\
1=2(\alpha\beta+\gamma\delta) & = & ,,\ \sqrt{ce}, \\
2=2(\alpha\gamma+\beta\delta) & = & ,,\ \sqrt{cd}, \\
3=2(\alpha\delta+\beta\gamma) & = & ,,\ \sqrt{be}, \\
4=\alpha^2-\beta^2+\gamma^2-\delta^2 & = & ,,\ \sqrt{ac}, \\
6=2(\alpha\gamma-\beta\delta) & = & ,,\ \sqrt{ab}, \\
8=\alpha^2+\beta^2-\gamma^2-\delta^2 & = & ,,\ \sqrt{bc}, \\
9=2(\alpha\beta-\gamma\delta) & = & ,,\ \sqrt{de}, \\
12=\alpha^2-\beta^2-\gamma^2-\delta^2 & = & ,,\ \sqrt{ad}, \\
15=2(\alpha\delta-\beta\gamma) & = & ,,\ \sqrt{ae}.
\end{array}$$

It hence appears that we can form an arrangement

$$\left|\begin{array}{ccc} c^2_{12}, & c^2_1, & c^2_6 \\ c^2_9, & -c^2_4, & c^2_3 \\ c^2_2, & -c^2_{15}, & -c^2_8 \end{array}\right| \div c^2_0 \quad = \left|\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right|$$

a system of coefficients in the transformation between two sets of rectangular coordinates.

We have between the squares of these coefficients of transformation 6+9 equations

$$a^2\ +b^2\ +c^2\ =1,$$
$$a'^2\ +b'^2\ +c'^2=1,$$
$$a''^2+b''^2+c''^2=1,$$

$$a^2\ +a'^2+a''^2=1,$$
$$b^2\ +b'^2\ +b''^2=1,$$
$$c^2\ +c'^2\ +c''^2=1,$$

$$b^2+c^2=a'^2+a''^2,\ b'^2+c'^2=a''^2+a^2,\ b''^2+c''^2=a^2+a'^2,$$
$$c^2+a^2=b'^2+b''^2,\ c'^2+a'^2=b''^2=b^2,\ c''^2+a''^2=b^2+b'^2,$$
$$a^2+b^2=c^2\ +c''^2,\ a'^2+b'^2=c''^2=c^2,\ a''^2+b''^2=c^2+c'^2,$$

that is

c^4	c^4	c^4	c^4
12	+ 1	+ 6	− 0
9	+ 4	+ 3	− 0
2	+15	+ 8	− 0
12	+ 9	+ 2	− 0
1	+ 4	+15	− 0
6	+ 3	+ 8	− 0
1	+ 6	− 9	− 2
6	+12	− 4	−15
12	+ 1	− 3	− 8
4	+ 3	− 2	−12
3	+ 9	−15	− 1
9	+ 4	− 8	− 6
15	+ 8	−12	− 9
8	+ 2	− 1	− 4
2	+15	− 6	− 3

=0 ;

and between the products a system of 6+9 equations

$$\begin{array}{l}
a'a''+b'b''+c'c''=0,\\
a''a+b''b+c''c=0,\\
aa'+bb'+cc'=0,\\
\\
bc+b'c'+b''c''=0,\\
ca+c'a'+c''a''=0,\\
ab+a'b'+a''b''=0,\\
\\
\begin{array}{llll}
a,\ b,\ c & =b'c''-b''c', & c'a''-c''a', & a'b''-a''b',\\
a',\ b',\ c' & b''c-bc'', & c''a-ca'', & ab''-a''b,\\
a'',\ b'',\ c'' & bc'-b'c, & ca'-c'a, & ab'-a'b,
\end{array}
\end{array}$$

that is

c^2	c^2	c^2	c^2	c^2	c^2
9	2	+ 4	15	− 3	8
2	12	−15	1	− 8	6
12	9	− 1	4	+ 6	3
1	6	− 4	3	+15	8
6	12	+ 3	9	− 8	2
12	1	− 9	4	− 2	15
−0	12	+ 4	8	+ 3	15
−0	1	+ 3	2	+ 8	9
−0	6	− 9	15	+ 2	4
−0	9	−15	6	+ 8	1
+0	4	− 8	12	− 6	2
−0	3	−12	15	− 1	2
−0	2	+ 1	3	+ 4	6
+0	15	+ 6	9	− 3	12
+0	8	−12	4	− 9	1

=0 ;

each of the first set of 15 giving a homogeneous linear relation between four terms c^4; and each of the second set giving a homogeneous linear relation between three terms $c^2.\ c^2$, formed with the 10 constants c. Thus the first equation is $c_{12}{}^4+c_1{}^4+c_6{}^4-c_0{}^4=0$; and so for the other lines of the two diagrams.

79. I form in the two notations the following tables:—

TABLE of the 16 KUMMER hexads.

A	A	A	A	A	B	B	B	B	C	C	C	D	D	E	A
B	C	D	E	F	C	D	E	F	D	E	F	E	F	F	B
AB	AC	AD	AE	AB	BC	BD	BE	AB	CD	CE	AC	DE	AD	AE	C
CD	BD	BC	BC	AC	AD	AC	AC	BC	AB	AB	BC	AB	BD	BE	D
CE	BE	BE	BD	AD	AE	AE	AD	BD	AE	AD	CD	AC	CD	CE	E
DE	DE	CE	CD	AE	DE	CE	CD	BE	BE	BD	CE	BC	DE	DE	F

=

11	11	11	11	11	7	7	7	7	5	5	5	13	13	14	11
7	5	13	14	10	5	13	14	10	13	14	10	14	10	10	7
6	4	14	12	6	8	0	3	6	2	1	4	9	12	15	5
2	0	8	8	4	12	4	4	8	6	6	8	6	0	3	13
1	3	3	3	12	15	15	12	0	15	12	2	4	2	1	14
9	9	1	2	15	9	1	2	3	3	0	1	8	9	9	10

80. Table of the 120 pairs.

A.B AC.BC AD.BD AE.BE	A.C AB.BC AD.CD AE.CE	A.D AB.BD AC.CD AE.DE	A.E AB.BE AC.CE AD.DE	A.F BC.DE BD.CE BE.CD	B.C AB.AC BD.CD BE.CE	B.D AB.AD BC.CD BE.DE	B.E AB.AE BC.CE BD.DE	B.F AC.DE AD.CE AE.CD	C.D AC.AD BC.BD CE.DE	C.E AC.AE BC.BE CD.DE	C.F AB.DE AD.BE AE.BD	D.E AD.AE BD.BE CD.CE	D.F AB.CE AC.BE AE.BC	E.F AB.CD AC.BD AD.BC
F.AB C.DE D.CE E.CD	F.AC B.DE D.BE E.BD	F.AD B.CE C.BE E.BC	F.AE B.CD C.BD D.BC	B.AB C.AC D.AD E.AE	F.BC A.DE D.AE E.AD	F.BD A.CE C.AE E.AC	F.BE A.CD C.AD D.AC	A.AB C.BC D.BD E.BE	F.CD A.BE B.AE E.AB	F.CE A.BD B.AD D.AB	A.AC B.BC D.CD E.CE	F.DE A.BC B.AC C.AB	A.AD B.BD C.CD E.DE	A.AE B.BE C.CE D.DE

=

11.7 4.8 12.0 15.3	11.5 6.8 12.2 15.1	11.13 6.0 4.2 15.9	11.14 6.3 4.1 12.9	11.10 8.9 0.1 3.2	7.5 6.4 0.2 3.1	7.13 6.12 8.2 3.9	7.14 6.15 8.1 0.9	7.10 4.9 12.1 15.2	5.13 4.12 8.0 1.9	5.14 4.15 8.3 2.9	5.10 6.9 12.3 15.0	13.14 12.15 0.3 2.1	13.10 6.1 4.3 15.8	14.10 6.2 4.0 12.8
10.6 5.9 13.1 14.2	10.4 7.9 13.3 14.0	10.12 7.1 5.3 14.8	10.15 7.2 5.0 13.8	7.6 5.4 13.12 14.15	10.8 11.9 13.15 14.12	10.0 11.1 5.15 14.4	10.3 11.2 5.12 13.4	11.6 5.8 13.0 14.3	10.2 11.3 7.15 14.6	10.3 11.0 7.12 13.6	11.4 7.8 13.2 14.1	10.9 11.8 7.4 5.6	11.12 7.0 5.2 14.9	11.15 7.3 5.1 13.9

81. TABLE of the 60 GÖPEL tetrads.

A . B . AE . BE	C . D . CE . DE	E . F . AB . CD	AC . BD . AD . BC
A . B . AD . BD	C . E . CD . DE	D . F . AB . CE	AC . BE . AE . BC
A . B . AC . BC	C . F . AB . DE	D . E . CD . CE	AD . BE . AE . BD
A . C . AE . CE	B . D . BE . DE	E . F . AC . BD	AB . CD . AD . BC
A . C . AD . CD	B . E . BD . DE	D . F . AC . BE	AB . CE . AE . BC
A . C . AB . BC	B . F . AC . DE	D . E . BD . BE	AD . CE . AE . CD
A . D . AE . DE	B . C . BE . CE	E . F . AD . BC	AB . CD . AC . BD
A . D . AC . CD	B . E . BC . CE	C . F . AD . BE	AB . DE . AE . BD
A . D . AB . BD	B . F . AD . CE	C . E . CD . DE	AC . DE . AE . CD
A . E . AD . DE	B . C . BD . CD	D . F . AE . BC	AB . CE . AC . BE
A . E . AC . CE	B . D . BC . CD	C . F . AE . BD	AB . DE . AD . BE
A . E . AB . BE	B . F . AE . CD	C . D . BC . BD	AC . DE . AD . CE
A . F . BC . DE	B . C . AB . AC	D . E . AD . AE	BD . CE . BE . CD
A . F . BD . CE	B . D . AB . AD	C . E . AC . AE	BC . DE . BE . CD
A . F . BE . CD	B . E . AB . AE	C . D . AC . AD	BC . DE . BD . CE

=

11 7 15 3	5 13 1 9	14 10 6 2	4 0 12 8
11 7 12 0	5 14 2 9	13 10 6 1	4 3 15 8
11 7 4 8	5 10 6 9	13 14 2 1	12 3 15 0
11 5 15 1	7 13 3 9	14 10 4 0	6 2 12 8
11 5 12 2	7 14 0 9	13 10 4 3	6 1 15 8
11 5 6 8	7 10 4 9	13 14 0 3	12 1 15 2
11 13 15 9	7 5 3 1	14 10 12 8	6 2 4 0
11 13 4 2	7 14 8 1	5 10 12 3	6 9 15 0
11 13 6 0	7 10 12 1	5 14 2 9	4 9 15 2
11 14 12 9	7 5 0 2	13 10 15 8	6 1 4 3
11 14 4 1	7 13 8 2	5 10 15 0	6 9 12 3
11 14 6 3	7 10 15 2	5 13 8 0	4 9 12 1
11 10 8 9	7 5 6 4	13 14 12 15	0 1 3 2
11 10 0 1	7 13 6 12	5 14 4 15	8 9 3 2
11 10 3 2	7 14 6 15	5 13 4 12	8 9 0 1

The product-theorem, and its results.

82. The product-theorem was

$$\vartheta\begin{pmatrix}\alpha,\beta\\ \gamma,\delta\end{pmatrix}(u+u').\vartheta\begin{pmatrix}\alpha',\beta'\\ \gamma',\delta'\end{pmatrix}(u-u')=\Sigma\Theta\begin{matrix}\tfrac{1}{2}(\alpha+\alpha')+p, & \tfrac{1}{2}(\beta+\beta')+q\\ \gamma+\gamma' & \delta+\delta\end{matrix}(2u).\Theta\begin{matrix}\tfrac{1}{2}(\alpha-\alpha')+p, & \tfrac{1}{2}(\beta-\beta')+q\\ \gamma-\gamma' & \delta-\delta'\end{matrix}(2u')$$

where only one argument is exhibited, viz. : $u+u'$, $u-u'$, $2u$, $2u'$ are written in place of $(u+u',\ v+v')$, $(u-u',\ v-v')$, $(2u,\ 2v)$, $(2u',\ 2v')$ respectively. The expression on the right hand side is always a sum of four terms, corresponding to the values (0, 0), (1, 0), (0, 1), and (1, 1) of (p, q). For the development of the results it was found convenient to use the following auxiliary diagram.

UPPER half of characteristic.

α	β	α'	β'	$\frac{1}{2}(\alpha+\alpha')$	$\frac{1}{2}(\beta+\beta')$	$\frac{1}{2}(\alpha-\alpha')$	$\frac{1}{2}(\beta-\beta')$	$\frac{1}{2}(\alpha+\alpha')+1$	$\frac{1}{2}(\beta+\beta')$	$\frac{1}{2}(\alpha-\alpha')+1$	$\frac{1}{2}(\beta-\beta')$	$\frac{1}{2}(\alpha+\alpha')$	$\frac{1}{2}(\beta+\beta')+1$	$\frac{1}{2}(\alpha-\alpha')$	$\frac{1}{2}(\beta-\beta')+1$	$\frac{1}{2}(\alpha+\alpha')+1$	$\frac{1}{2}(\beta+\beta')+1$	$\frac{1}{2}(\alpha-\alpha')+1$	$\frac{1}{2}(\beta-\beta')+1$
0	0	0	0	0	0	0	0	1	0	1	0	0	1	0	1	1	1	1	1
1	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{3}{2}$	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{3}{2}$	1	$\frac{3}{2}$	1
0	1	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{3}{2}$	0	$\frac{3}{2}$	1	$\frac{3}{2}$	1	$\frac{3}{2}$
1	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
0	0	1	0	$\frac{1}{2}$	0	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	1	$\frac{3}{2}$	1	$\frac{1}{2}$	1
1	0	1	0	1	0	0	0	0	0	1	0	1	1	0	1	0	1	1	1
0	1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
1	1	1	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{3}{2}$	0	$\frac{3}{2}$	0	$\frac{3}{2}$	1	$\frac{3}{2}$
0	0	0	1	0	$\frac{1}{2}$	0	$\frac{3}{2}$	1	$\frac{1}{2}$	1	$\frac{3}{2}$	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	1	$\frac{1}{2}$
1	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
0	1	0	1	0	1	0	0	1	1	1	0	0	0	0	1	1	0	1	1
1	1	0	1	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{3}{2}$	1	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	0	$\frac{3}{2}$	1
0	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0	1	1	1	$\frac{1}{2}$	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	1	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$\frac{3}{2}$	1	$\frac{1}{2}$
0	1	1	1	$\frac{1}{2}$	1	$\frac{3}{2}$	0	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{3}{2}$	1	$\frac{3}{2}$	0	$\frac{1}{2}$	1
1	1	1	1	1	1	0	0	0	1	1	0	1	0	0	1	0	0	1	1

LOWER half of characteristic.

γ	δ	γ'	δ'	$\gamma+\gamma'$	$\delta+\delta'$	$\gamma-\gamma'$	$\delta-\delta'$	$\gamma+\gamma'$	$\delta+\delta'$	$\gamma-\gamma'$	$\delta-\delta'$	$\gamma+\gamma'$	$\delta+\delta'$	$\gamma-\gamma'$	$\delta-\delta'$	$\gamma+\gamma'$	$\delta+\delta'$	$\gamma-\gamma'$	$\delta-\delta'$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
0	1	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	0	1	0	1	0	−1	0	1	0	−1	0	1	0	−1	0	1	0	−1	0
1	0	1	0	2	0	0	0	2	0	0	0	2	0	0	0	2	0	0	0
0	1	1	0	1	1	−1	1	1	1	−1	1	1	1	−1	1	1	1	−1	1
1	1	1	0	2	1	0	1	2	1	0	1	2	1	0	1	2	1	0	1
0	0	0	1	0	1	0	−1	0	1	0	−1	0	1	0	−1	0	1	0	−1
1	0	0	1	1	1	1	−1	1	1	1	−1	1	1	1	−1	1	1	1	−1
0	1	0	1	0	2	0	0	0	2	0	0	0	2	0	0	0	2	0	0
1	1	0	1	1	2	1	0	1	2	1	0	1	2	1	0	1	2	1	0
0	0	1	1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1
1	0	1	1	2	1	0	−1	2	1	0	−1	2	1	0	−1	2	1	0	−1
0	1	1	1	1	2	−1	0	0	2	−1	0	0	2	−1	0	1	2	−1	0
1	1	1	1	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0

83. The upper characters of the Θ's have thus the values 0, 1, $\frac{1}{2}$, $\frac{3}{2}$; the lower characters are originally 2, 1, 0, or -1, and these have when necessary to be by the addition or subtraction of 2 reduced to 0 or 1; the effect of this change is either to leave the Θ unaltered, or to multiply it by -1 or $\pm i$, as follows

$$\begin{array}{l|l} \Theta^{0}_{\gamma\pm 2} = \Theta^{0}_{\gamma}, & \Theta^{\frac{1}{2}}_{\gamma+2} = i\Theta^{\frac{1}{2}}_{\gamma},\ \ \Theta^{\frac{1}{2}}_{\gamma-2} = -i\Theta^{\frac{1}{2}}_{\gamma}, \\ \Theta^{1}_{\gamma\pm 2} = -\Theta^{1}_{\gamma}, & \Theta^{\frac{3}{2}}_{\gamma+2} = -i\Theta^{\frac{3}{2}}_{\gamma},\ \ \Theta^{\frac{3}{2}}_{\gamma+2} = i\Theta^{\frac{3}{2}}_{\gamma}, \end{array}$$

where only the first column of characters is shown, but the same rule applies to the second column; and where we must of course combine the multipliers corresponding to the first and second columns respectively: for instance

$$\Theta^{\frac{3}{2}\ \frac{1}{2}}_{\gamma+2\ \delta+2} = (-i\,.-i=)-\Theta^{\frac{3}{2}\ \frac{1}{2}}_{\gamma\ \ \delta}.$$

Thus taking the tenth line of the upper half, and the fifth line of the lower half, we have

$$\begin{array}{c|c|cccc|cccc|cccc|cccc} 10 & 01 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} \\ \hline 00 & 10 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \end{array}$$

giving the value of $\vartheta^{1\ 0}_{0\ 0}(u+u').\vartheta^{0\ 1}_{1\ 0}(u-u')$: viz. this is

$$\begin{array}{ll} = \ \ \Theta^{\frac{1}{2}\ \frac{1}{2}}_{1\ 0}(2u)\,.\,\Theta^{\frac{1}{2}\ \frac{3}{2}}_{-1\ 0}(2u') = & i\Theta^{\frac{1}{2}\ \frac{1}{2}}_{1\ 0}(2u)\,.\,\Theta^{\frac{1}{2}\ \frac{3}{2}}_{1\ 0}(2u') \\ \ \ \ +\Theta^{\frac{3}{2}\ \frac{1}{2}}_{1\ 0}(\ ,,\ )\,.\,\Theta^{\frac{3}{2}\ \frac{3}{2}}_{-1\ 0}(\ ,,\ ) & -i\Theta^{\frac{3}{2}\ \frac{1}{2}}_{1\ 0}(\ ,,\ )\,.\,\Theta^{\frac{3}{2}\ \frac{3}{2}}_{1\ 0}(\ ,,\ ) \\ \ \ \ +\Theta^{\frac{1}{2}\ \frac{3}{2}}_{1\ 0}(\ ,,\ )\,.\,\Theta^{\frac{1}{2}\ \frac{1}{2}}_{-1\ 0}(\ ,,\ ) & +i\Theta^{\frac{1}{2}\ \frac{3}{2}}_{1\ 0}(\ ,,\ )\,.\,\Theta^{\frac{1}{2}\ \frac{1}{2}}_{1\ 0}(\ ,,\ ) \\ \ \ \ +\Theta^{\frac{3}{2}\ \frac{3}{2}}_{1\ 0}(\ ,,\ )\,.\,\Theta^{\frac{3}{2}\ \frac{1}{2}}_{-1\ 0}(\ ,,\ ) & -i\Theta^{\frac{3}{2}\ \frac{3}{2}}_{1\ 0}(\ ,,\ )\,.\,\Theta^{\frac{1}{2}\ \frac{1}{2}}_{1\ 0}(\ ,,\ ), \end{array}$$

where the first column is the value given directly by the diagram, and which is then reduced to that given by the second column.

84. But instead of the Θ's we introduce single letters (X, Y, Z, W), (E, F, G, H), (I, J, K, L), (M, N, P, Q), with the suffixes (0, 1, 2, 3), in all 64 symbols, thus

$$\Theta\begin{array}{c|cccc} & 00 & 10 & 01 & 11 \\ \hline 00 & & & & \\ 10 & & & & \\ 01 & & & & \\ 11 & & & & \end{array}(2u) = \begin{array}{c|cccc} & X & Y & Z & W \\ \hline 0 & & & & \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \end{array}$$

that is $\Theta^{00}_{00}(2u)=X$, $\Theta^{10}_{00}=Y$, &c.

$\Theta^{00}_{10}(2u)=X_1$ . .

$\vdots$

$$\Theta\begin{array}{c|cccc} & \frac{1}{2}0 & \frac{1}{2}1 & \frac{3}{2}0 & \frac{3}{2}1 \\ \hline 00 & & & & \\ 10 & & & & \\ 01 & & & & \\ 11 & & & & \end{array}(2u) = \begin{array}{c|cccc} & E & F & G & H \\ \hline 0 & & & & \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \end{array}$$

that is $\Theta^{\frac{1}{2}0}_{00}(2u)=E$, &c.

$$\Theta\begin{array}{c|cccc} & 0\frac{1}{2} & 1\frac{1}{2} & 0\frac{3}{2} & 1\frac{3}{2} \\ \hline 00 & & & & \\ 10 & & & & \\ 01 & & & & \\ 11 & & & & \end{array}(2u) = \begin{array}{c|cccc} & I & J & K & L \\ \hline 0 & & & & \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \end{array}$$

The functions of $(2u')$ are denoted in like manner by accented letters

$\Theta^{00}_{00}(2u')=X'$, &c.

$$\Theta\begin{array}{c|cccc} & \frac{1}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} & \frac{1}{2}\frac{3}{2} & \frac{3}{2}\frac{3}{2} \\ \hline 00 & & & & \\ 10 & & & & \\ 01 & & & & \\ 11 & & & & \end{array}(2u) = \begin{array}{c|cccc} & M & N & P & Q \\ \hline 0 & & & & \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \end{array}$$

85. To simplify the expression of the results, instead of in each case writing down the suffixes, I have indicated them by means of the column headed "Suff."

Thus

$$| \, 8-0 \, | \qquad \vartheta^{01}_{01}u+u' \, . \, \vartheta^{00}_{00}u-u' = XX' + YY' + ZZ' + WW' \qquad \begin{array}{c}\text{Suff.}\\ | \, 2 \, |\end{array}$$

means that the equation is to be read

$$=X_2X_2'+Y_2Y_2'+Z_2Z_2'+W_2W_2'.$$

It is hardly necessary to mention that the $| \, 8-0 \, |$ of the left hand column shows the current numbers of the theta-functions; viz.: the left hand side of the equation is $\vartheta_8(u+u') \, . \, \vartheta_0(u-u')$.

And by a preceding remark the single arguments $u+u'$ and $u-u'$ are written in place of $(u+u', v+v')$ and $(u-u', v-v')$ respectively.

The 256 equations now are

86. FIRST set, 64 equations.

								Suffixes.
0—0	$\vartheta^{00}_{00}\,u+u'\,.$	$\vartheta^{00}_{00}\,u-u'$	$=$	XX'	$+YY'$	$+ZZ'$	$+WW'$	0
4—0	$^{00}_{10}$	$^{00}_{00}$	$=$	XX'	$+YY'$	$+ZZ'$	$+WW'$	1
8—0	$^{00}_{01}$	$^{00}_{00}$	$=$	XX'	$+YY'$	$+ZZ'$	$+WW'$	2
12—0	$^{00}_{11}$	$^{00}_{00}$	$=$	XX'	$+YY'$	$+ZZ'$	$+WW'$	3
0—4	$\vartheta^{00}_{00}\,u+u'\,.$	$\vartheta^{00}_{10}\,u-u'$	$=$	XX'	$-YY'$	$+ZZ'$	$-WW'$	1
4—4	$^{00}_{10}$	$^{00}_{10}$	$=$	XX'	$-YY'$	$+ZZ'$	$-WW'$	0
8—4	$^{00}_{01}$	$^{00}_{10}$	$=$	XX'	$-YY'$	$+ZZ'$	$-WW'$	3
12—4	$^{00}_{11}$	$^{00}_{10}$	$=$	XX'	$-YY'$	$+ZZ'$	$-WW'$	2
0—8	$\vartheta^{00}_{00}\,u+u'\,.$	$\vartheta^{00}_{01}\,u-u'$	$=$	XX'	$+YY'$	$-ZZ'$	$-WW'$	2
4—8	$^{00}_{10}$	$^{00}_{01}$	$=$	XX'	$+YY'$	$-ZZ'$	$-WW'$	3
8—8	$^{00}_{01}$	$^{00}_{01}$	$=$	XX'	$+YY'$	$-ZZ'$	$-WW'$	0
12—8	$^{00}_{11}$	$^{00}_{01}$	$=$	XX'	$+YY'$	$-ZZ'$	$-WW'$	1
0—12	$\vartheta^{00}_{00}\,u+u'\,.$	$\vartheta^{00}_{11}\,u-u'$	$=$	XX'	$-YY'$	$-ZZ'$	$+WW'$	3
4—12	$^{00}_{10}$	$^{00}_{11}$	$=$	XX'	$-YY'$	$-ZZ'$	$+WW'$	2
8—12	$^{00}_{01}$	$^{00}_{11}$	$=$	XX'	$-YY'$	$-ZZ'$	$+WW'$	1
12—12	$^{00}_{11}$	$^{00}_{11}$	$=$	XX'	$-YY'$	$-ZZ'$	$+WW'$	0

FIRST set, 64 equations (continued).

				Suffixes.
1—1	$\vartheta^{10}_{00}\,u+u'\,.$	$\vartheta^{10}_{00}\,u-u'$	$= YX' + XY' + WZ' + ZW'$	0
5—1	$^{10}_{10}$	$^{10}_{00}$	$= YX' + XY' + WZ' + ZW'$	1
9—1	$^{10}_{01}$	$^{10}_{00}$	$= YX' + XY' + WZ' + ZW'$	2
13—1	$^{10}_{11}$	$^{10}_{00}$	$= YX' + XY' + WZ' + ZW'$	3
1—5	$\vartheta^{10}_{00}\,u+u'\,.$	$\vartheta^{10}_{10}\,u-u'$	$= YX' - XY' + WZ' - ZW'$	1
5—5	$^{10}_{10}$	$^{10}_{10}$	$= -YX' + XY' - WZ' + ZW'$	0
9—5	$^{10}_{01}$	$^{10}_{10}$	$= YX' - XY' + WZ' - ZW'$	3
13—5	$^{10}_{11}$	$^{10}_{10}$	$= -YX' + XY' - WZ' + ZW'$	2
1—9	$\vartheta^{10}_{00}\,u+u'\,.$	$\vartheta^{10}_{01}\,u-u'$	$= YX' + XY' - WZ' - ZW'$	2
5—9	$^{10}_{10}$	$^{10}_{01}$	$= YX' + XY' - WZ' - ZW'$	3
9—9	$^{10}_{01}$	$^{10}_{01}$	$= YX' + XY' - WZ' - ZW'$	0
13—9	$^{10}_{11}$	$^{10}_{01}$	$= YX' + XY' - WZ' - ZW'$	1
1—13	$\vartheta^{10}_{00}\,u+u'\,.$	$\vartheta^{10}_{11}\,u-u'$	$= YX' - XY' - WZ' + ZW'$	3
5—13	$^{10}_{10}$	$^{10}_{11}$	$= -YX' + XY' + WZ' - ZW'$	2
9—13	$^{10}_{01}$	$^{10}_{11}$	$= YX' - XY' - WZ' + ZW'$	1
13—13	$^{10}_{11}$	$^{10}_{11}$	$= -YX' + XY' + WZ' - ZW'$	0

First set, 64 equations (continued).

				Suffixes.
2—2	$\vartheta^{01}_{00}\,u+u'$.	$\vartheta^{01}_{00}\,u-u'$	$= ZX' + WY' + XZ' + YW'$	0
6—2	$\substack{01\\10}$	$\substack{01\\00}$	$= ZX' + WY' + XZ' + YW'$	1
10—2	$\substack{01\\01}$	$\substack{01\\00}$	$= ZX' + WY' + XZ' + YW'$	2
14—2	$\substack{01\\11}$	$\substack{01\\00}$	$= ZX' + WY' + XZ' + YW'$	3
2—6	$\vartheta^{01}_{00}\,u+u'$.	$\vartheta^{01}_{10}\,u-u'$	$= ZX' - WY' + XZ' - YW'$	1
6—6	$\substack{01\\10}$	$\substack{01\\10}$	$= ZX' - WY' + XZ' - YW'$	0
10—6	$\substack{01\\01}$	$\substack{01\\10}$	$= ZX' - WY' + XZ' - YW'$	3
14—6	$\substack{01\\11}$	$\substack{01\\10}$	$= ZX' - WY' + XZ' - YW'$	2
2—10	$\vartheta^{01}_{00}\,u+u'$.	$\vartheta^{01}_{01}\,u-u'$	$= ZX' + WY' - XZ' - YW'$	2
6—10	$\substack{01\\10}$	$\substack{01\\01}$	$= ZX' + WY' - XZ' - YW'$	3
10—10	$\substack{01\\01}$	$\substack{01\\01}$	$= -ZX' - WY' + XZ' + YW'$	0
14—10	$\substack{01\\11}$	$\substack{01\\01}$	$= -ZX' - WY' + XZ' + YW'$	1
2—14	$\vartheta^{01}_{00}\,u+u'$.	$\vartheta^{01}_{11}\,u-u'$	$= ZX' - WY' - XZ' + YW'$	3
6—14	$\substack{01\\10}$	$\substack{01\\11}$	$= ZX' - WY' - XZ' + YW'$	2
10—14	$\substack{01\\01}$	$\substack{01\\11}$	$= -ZX' + WY' + XZ' - YW'$	1
14—14	$\substack{01\\11}$	$\substack{01\\11}$	$= -ZX' + WY' + XZ' - YW'$	0

FIRST set, 64 equations (concluded).

							Suffixes.
3–3	$\vartheta^{11}_{00}\,u+u'$.	$\vartheta^{11}_{00}\,u-u'$	$=\ \mathrm{W\,X'}$	$+\mathrm{Z\,Y'}$	$+\mathrm{Y\,Z'}$	$+\mathrm{X\,W'}$	0
7–3	$^{11}_{10}$	$^{11}_{00}$	$=\ \mathrm{W\,X'}$	$+\mathrm{Z\,Y'}$	$+\mathrm{Y\,Z'}$	$+\mathrm{X\,W'}$	1
11–3	$^{11}_{01}$	$^{11}_{00}$	$=\ \mathrm{W\,X'}$	$+\mathrm{Z\,Y'}$	$+\mathrm{Y\,Z'}$	$+\mathrm{X\,W'}$	2
15–3	$^{11}_{11}$	$^{11}_{00}$	$=\ \mathrm{W\,X'}$	$+\mathrm{Z\,Y'}$	$+\mathrm{Y\,Z'}$	$+\mathrm{X\,W'}$	3
3–7	$\vartheta^{11}_{00}\,u+u'$.	$\vartheta^{11}_{10}\,u-u'$	$=\ \mathrm{W\,X'}$	$-\mathrm{Z\,Y'}$	$+\mathrm{Y\,Z'}$	$-\mathrm{X\,W'}$	1
7–7	$^{11}_{10}$	$^{11}_{10}$	$=-\mathrm{W\,X'}$	$+\mathrm{Z\,Y'}$	$-\mathrm{Y\,Z'}$	$+\mathrm{X\,W'}$	0
11–7	$^{11}_{01}$	$^{11}_{10}$	$=\ \mathrm{W\,X'}$	$-\mathrm{Z\,Y'}$	$+\mathrm{Y\,Z'}$	$-\mathrm{X\,W'}$	3
15–7	$^{11}_{11}$	$^{11}_{10}$	$=-\mathrm{W\,X'}$	$+\mathrm{Z\,Y'}$	$-\mathrm{Y\,Z'}$	$+\mathrm{X\,W'}$	2
3–11	$\vartheta^{11}_{00}\,u+u'$.	$\vartheta^{11}_{01}\,u-u'$	$=\ \mathrm{W\,X'}$	$+\mathrm{Z\,Y'}$	$-\mathrm{Y\,Z'}$	$-\mathrm{X\,W'}$	2
7–11	$^{11}_{10}$	$^{11}_{01}$	$=\ \mathrm{W\,X'}$	$+\mathrm{Z\,Y'}$	$-\mathrm{Y\,Z'}$	$-\mathrm{X\,W'}$	3
11–11	$^{11}_{01}$	$^{11}_{01}$	$=-\mathrm{W\,X'}$	$-\mathrm{Z\,Y'}$	$+\mathrm{Y\,Z'}$	$+\mathrm{X\,W'}$	0
15–11	$^{11}_{11}$	$^{11}_{01}$	$=-\mathrm{W\,X'}$	$-\mathrm{Z\,Y'}$	$+\mathrm{Y\,Z'}$	$+\mathrm{X\,W'}$	1
3–15	$\vartheta^{11}_{00}\,u+u'$.	$\vartheta^{11}_{11}\,u-u'$	$=\ \mathrm{W\,X'}$	$-\mathrm{Z\,Y'}$	$-\mathrm{Y\,Z'}$	$+\mathrm{X\,W'}$	3
7–15	$^{11}_{10}$	$^{11}_{11}$	$=-\mathrm{W\,X'}$	$+\mathrm{Z\,Y'}$	$+\mathrm{Y\,Z'}$	$-\mathrm{X\,W'}$	2
11–15	$^{11}_{01}$	$^{11}_{11}$	$=-\mathrm{W\,X'}$	$+\mathrm{Z\,Y'}$	$+\mathrm{Y\,Z'}$	$-\mathrm{X\,W'}$	1
15–15	$^{11}_{11}$	$^{11}_{11}$	$=\ \mathrm{W\,X'}$	$-\mathrm{Z\,Y'}$	$-\mathrm{Y\,Z'}$	$+\mathrm{X\,W'}$	0

87. SECOND set, 64 equations.

				Suffixes.
1−0	$\vartheta^{10}_{00}\, u+u'$.	$\vartheta^{00}_{00}\, u-u'$	$= EE' + GG' + FF' + HH'$	0
5−0	$^{10}_{10}$	$^{00}_{00}$	$= EE' + GG' + FF' + HH'$	1
9−0	$^{10}_{01}$	$^{00}_{00}$	$= EE' + GG' + FF' + HH'$	2
13−0	$^{10}_{11}$	$^{00}_{00}$	$= EE' + GG' + FF' + HH'$	3
1−4	$\vartheta^{10}_{00}\, u+u'$.	$\vartheta^{00}_{10}\, u-u'$	$= -iEE' + iGG' - iFF' + iHH'$	1
5−4	$^{10}_{10}$	$^{00}_{10}$	$= iEE' - iGG' + iFF' - iHH'$	0
9−4	$^{10}_{01}$	$^{00}_{10}$	$= -iEE' + iGG' - iFF' + iHH'$	3
13−4	$^{10}_{11}$	$^{00}_{10}$	$= iEE' - iGG' + iFF' - iHH'$	2
1−8	$\vartheta^{10}_{00}\, u+u'$.	$\vartheta^{00}_{01}\, u-u'$	$= EE' + GG' - FF' - HH'$	2
5−8	$^{10}_{10}$	$^{00}_{01}$	$= EE' + GG' - FF' - HH'$	3
9−8	$^{10}_{01}$	$^{00}_{01}$	$= EE' + GG' - FF' - HH'$	0
13−8	$^{10}_{11}$	$^{00}_{01}$	$= EE' + GG' - FF' - HH'$	1
1−12	$\vartheta^{10}_{00}\, u+u'$.	$\vartheta^{00}_{11}\, u-u'$	$= -iEE' + iGG' + iFF' - iHH'$	3
5−12	$^{10}_{10}$	$^{00}_{11}$	$= iEE' - iGG' - iFF' + iHH'$	2
9−12	$^{10}_{01}$	$^{00}_{11}$	$= -iEE' + iGG' + iFF' - iHH'$	1
13−12	$^{10}_{11}$	$^{00}_{11}$	$= iEE' - iGG' - iFF' + iHH'$	0

SECOND set, 64 equations (continued).

					Suffixes.
0−1	$\vartheta^{00}_{00}\, u+u'$.	$\vartheta^{10}_{00}\, u-u'$	=	$\text{E G}' + \text{G E}' + \text{F H}' + \text{H F}'$	0
4−1	$^{00}_{10}$	$^{10}_{00}$	=	$\text{E G}' + \text{G E}' + \text{F H}' + \text{H F}'$	1
8−1	$^{00}_{01}$	$^{10}_{00}$	=	$\text{E G}' + \text{G E}' + \text{F H}' + \text{H F}'$	2
12−1	$^{00}_{11}$	$^{10}_{00}$	=	$\text{E G}' + \text{G E}' + \text{F H}' + \text{H F}'$	3
0−5	$\vartheta^{00}_{00}\, u+u'$.	$\vartheta^{10}_{10}\, u-u'$	=	$i\text{E G}' - i\text{G E}' + i\text{F H}' - i\text{H F}'$	1
4−5	$^{00}_{10}$	$^{10}_{10}$	=	$i\text{E G}' - i\text{G E}' + i\text{F H}' - i\text{H F}'$	0
8−5	$^{00}_{01}$	$^{10}_{10}$	=	$i\text{E G}' - i\text{G E}' + i\text{F H}' - i\text{H F}'$	3
12−5	$^{00}_{11}$	$^{10}_{10}$	=	$i\text{E G}' - i\text{G E}' + i\text{F H}' - i\text{H F}'$	2
0−9	$\vartheta^{00}_{00}\, u+u'$.	$\vartheta^{10}_{01}\, u-u'$	=	$\text{E G}' + \text{G E}' - \text{F H}' - \text{H F}'$	2
4−9	$^{00}_{10}$	$^{10}_{01}$	=	$\text{E G}' + \text{G E}' - \text{F H}' - \text{H F}'$	3
8−9	$^{00}_{01}$	$^{10}_{01}$	=	$\text{E G}' + \text{G E}' - \text{F H}' - \text{H F}'$	0
12−9	$^{00}_{11}$	$^{10}_{01}$	=	$\text{E G}' + \text{G E}' - \text{F H}' - \text{H F}'$	1
0−13	$\vartheta^{00}_{00}\, u+u'$.	$\vartheta^{10}_{11}\, u-u'$	=	$i\text{E G}' - i\text{G E}' - i\text{F H}' + i\text{H F}'$	3
4−13	$^{00}_{10}$	$^{10}_{11}$	=	$i\text{E G}' - i\text{G E}' - i\text{F H}' + i\text{H F}'$	2
8−13	$^{00}_{01}$	$^{10}_{11}$	=	$i\text{E G}' - i\text{G E}' - i\text{F H}' + i\text{H F}'$	1
12−13	$^{00}_{11}$	$^{10}_{11}$	=	$i\text{E G}' - i\text{G E}' - i\text{F H}' + i\text{H F}'$	0

SECOND set, 64 equations (continued).

				Suffixes.
3−2	$\vartheta^{11}_{00}\,u+u'\,.$	$\vartheta^{01}_{00}\,u-u'$	$= FE' + HG' + EF' + GH'$	0
7−2	$\begin{matrix}11\\10\end{matrix}$	$\begin{matrix}01\\00\end{matrix}$	$= FE' + HG' + EF' + GH'$	1
11−2	$\begin{matrix}11\\01\end{matrix}$	$\begin{matrix}01\\00\end{matrix}$	$= FE' + HG' + EF' + GH'$	2
15−2	$\begin{matrix}11\\11\end{matrix}$	$\begin{matrix}01\\00\end{matrix}$	$= FE' + HG' + EF' + GH'$	3
3−6	$\vartheta^{11}_{00}\,u+u'\,.$	$\vartheta^{01}_{10}\,u-u'$	$= -iFE' + iHG' - iEF' + iGH'$	1
7−6	$\begin{matrix}11\\10\end{matrix}$	$\begin{matrix}01\\10\end{matrix}$	$= iFE' - iHG' + iEF' - iGH'$	0
11−6	$\begin{matrix}11\\01\end{matrix}$	$\begin{matrix}01\\10\end{matrix}$	$= -iFE' + iHG' - iEF' + iGH'$	3
15−6	$\begin{matrix}11\\11\end{matrix}$	$\begin{matrix}01\\10\end{matrix}$	$= iFE' - iHG' + iEF' - iGH'$	2
3−10	$\vartheta^{11}_{00}\,u+u'\,.$	$\vartheta^{01}_{01}\,u-u'$	$= FE' + HG' - EF' - GH'$	2
7−10	$\begin{matrix}11\\10\end{matrix}$	$\begin{matrix}01\\01\end{matrix}$	$= FE' + HG' - EF' - GH'$	3
11−10	$\begin{matrix}11\\01\end{matrix}$	$\begin{matrix}01\\01\end{matrix}$	$= -FE' - HG' + EF' + GH'$	0
15−10	$\begin{matrix}11\\11\end{matrix}$	$\begin{matrix}01\\01\end{matrix}$	$= -FE' - HG' + EF' + GH'$	1
3−14	$\vartheta^{11}_{00}\,u+u'\,.$	$\vartheta^{01}_{11}\,u-u'$	$= -iFE' + iHG' + iEF' - iGH'$	3
7−14	$\begin{matrix}11\\10\end{matrix}$	$\begin{matrix}01\\11\end{matrix}$	$= iFE' - iHG' - iEF' + iGH'$	2
11−14	$\begin{matrix}11\\01\end{matrix}$	$\begin{matrix}01\\11\end{matrix}$	$= iFE' - iHG' - iEF' + iGH'$	1
15−14	$\begin{matrix}11\\11\end{matrix}$	$\begin{matrix}01\\11\end{matrix}$	$= -iFE' + iHG' + iEF' - iGH'$	0

SECOND set, 64 equations (concluded).

				Suffixes.
2—3	$\vartheta^{01}_{00}\, u+u'$.	$\vartheta^{11}_{00}\, u-u'$	$= FG' + HE' + EH' + GF'$	0
6—3	$^{01}_{10}$	$^{11}_{00}$	$= FG' + HE' + EH' + GF'$	1
10—3	$^{01}_{01}$	$^{11}_{00}$	$= FG' + HE' + EH' + GF'$	2
14—3	$^{01}_{11}$	$^{11}_{00}$	$= FG' + HE' + EH' + GF'$	3
2—7	$\vartheta^{01}_{00}\, u+u'$.	$\vartheta^{11}_{10}\, u-u'$	$= iFG' - iHE' + iEH' - iGF'$	1
6—7	$^{01}_{10}$	$^{11}_{10}$	$= iFG' - iHE' + iEH' - iGF'$	0
10—7	$^{01}_{01}$	$^{11}_{10}$	$= iFG' - iHE' + iEH' - iGF'$	3
14—7	$^{01}_{11}$	$^{11}_{10}$	$= iFG' - iHE' + iEH' - iGF'$	2
2—11	$\vartheta^{01}_{00}\, u+u'$.	$\vartheta^{11}_{01}\, u-u'$	$= FG' + HE' - EH' - GF'$	2
6—11	$^{01}_{10}$	$^{11}_{01}$	$= FG' + HE' - EH' - GF'$	3
10—11	$^{01}_{01}$	$^{11}_{01}$	$= - FG' - HE' + EH' + GF'$	0
14—11	$^{01}_{11}$	$^{11}_{01}$	$= - FG' - HE' + EH' + GF'$	1
2—15	$\vartheta^{01}_{00}\, u+u'$.	$\vartheta^{11}_{11}\, u-u'$	$= iFG' - iHE' - iEH' + iGF'$	3
6—15	$^{01}_{10}$	$^{11}_{11}$	$= iFG' - iHE' - iEH' + iGF'$	2
10—15	$^{01}_{01}$	$^{11}_{11}$	$= -iFG' + iHE' + iEH' - iGF'$	1
14—15	$^{01}_{11}$	$^{11}_{11}$	$= -iFG' + iHE' + iEH' - iGF'$	0

88. THIRD set, 64 equations.

				Suffixes.
2—0	$\vartheta^{01}_{00}\,u+u'$	$.\,\vartheta^{00}_{00}\,u-u'$	$= \mathrm{I}\,\mathrm{I}' + \mathrm{J}\,\mathrm{J}' + \mathrm{K}\,\mathrm{K}' + \mathrm{L}\,\mathrm{L}'$	0
6—0	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}00\\00\end{smallmatrix}$	$= \mathrm{I}\,\mathrm{I}' + \mathrm{J}\,\mathrm{J}' + \mathrm{K}\,\mathrm{K}' + \mathrm{L}\,\mathrm{L}'$	1
10—0	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}00\\00\end{smallmatrix}$	$= \mathrm{I}\,\mathrm{I}' + \mathrm{J}\,\mathrm{J}' + \mathrm{K}\,\mathrm{K}' + \mathrm{L}\,\mathrm{L}'$	2
14—0	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}00\\00\end{smallmatrix}$	$= \mathrm{I}\,\mathrm{I}' + \mathrm{J}\,\mathrm{J}' + \mathrm{K}\,\mathrm{K}' + \mathrm{L}\,\mathrm{L}'$	3
2—4	$\vartheta^{01}_{00}\,u+u'$	$.\,\vartheta^{00}_{10}\,u-u'$	$= \mathrm{I}\,\mathrm{I}' - \mathrm{J}\,\mathrm{J}' + \mathrm{K}\,\mathrm{K}' - \mathrm{L}\,\mathrm{L}'$	1
6—4	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}00\\10\end{smallmatrix}$	$= \mathrm{I}\,\mathrm{I}' - \mathrm{J}\,\mathrm{J}' + \mathrm{K}\,\mathrm{K}' - \mathrm{L}\,\mathrm{L}'$	0
10—4	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}00\\10\end{smallmatrix}$	$= \mathrm{I}\,\mathrm{I}' - \mathrm{J}\,\mathrm{J}' + \mathrm{K}\,\mathrm{K}' - \mathrm{L}\,\mathrm{L}'$	3
14—4	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}00\\10\end{smallmatrix}$	$= \mathrm{I}\,\mathrm{I}' - \mathrm{J}\,\mathrm{J}' + \mathrm{K}\,\mathrm{K}' - \mathrm{L}\,\mathrm{L}'$	2
2—8	$\vartheta^{01}_{00}\,u+u'$	$.\,\vartheta^{00}_{01}\,u-u'$	$= -i\mathrm{I}\,\mathrm{I}' - i\mathrm{J}\,\mathrm{J}' + i\mathrm{K}\,\mathrm{K}' + i\mathrm{L}\,\mathrm{L}'$	2
6—8	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}00\\01\end{smallmatrix}$	$= -i\mathrm{I}\,\mathrm{I}' - i\mathrm{J}\,\mathrm{J}' + i\mathrm{K}\,\mathrm{K}' + i\mathrm{L}\,\mathrm{L}'$	3
10—8	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}00\\01\end{smallmatrix}$	$= i\mathrm{I}\,\mathrm{I}' + i\mathrm{J}\,\mathrm{J}' - i\mathrm{K}\,\mathrm{K}' - i\mathrm{L}\,\mathrm{L}'$	0
14—8	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}00\\01\end{smallmatrix}$	$= i\mathrm{I}\,\mathrm{I}' + i\mathrm{J}\,\mathrm{J}' - i\mathrm{K}\,\mathrm{K}' - i\mathrm{L}\,\mathrm{L}'$	1
2—12	$\vartheta^{01}_{00}\,u+u'$	$.\,\vartheta^{00}_{11}\,u-u'$	$= -i\mathrm{I}\,\mathrm{I}' + i\mathrm{J}\,\mathrm{J}' + i\mathrm{K}\,\mathrm{K}' - i\mathrm{L}\,\mathrm{L}'$	3
6—12	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}00\\11\end{smallmatrix}$	$= -i\mathrm{I}\,\mathrm{I}' + i\mathrm{J}\,\mathrm{J}' + i\mathrm{K}\,\mathrm{K}' - i\mathrm{L}\,\mathrm{L}'$	2
10—12	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}00\\11\end{smallmatrix}$	$= i\mathrm{I}\,\mathrm{I}' - i\mathrm{J}\,\mathrm{J}' - i\mathrm{K}\,\mathrm{K}' + i\mathrm{L}\,\mathrm{L}'$	1
14—12	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}00\\11\end{smallmatrix}$	$= i\mathrm{I}\,\mathrm{I}' - i\mathrm{J}\,\mathrm{J}' - i\mathrm{K}\,\mathrm{K}' + i\mathrm{L}\,\mathrm{L}'$	0

THIRD set, 64 equations (continued).

				Suffixes.
3−1	$\vartheta^{11}_{00}\,u+u'\,.$	$\vartheta^{10}_{00}\,u-u'$	$= \mathrm{J\,I'} + \mathrm{I\,J'} + \mathrm{L\,K'} + \mathrm{K\,L'}$	0
7−1	$\begin{matrix}11\\10\end{matrix}$	$\begin{matrix}10\\00\end{matrix}$	$= \mathrm{J\,I'} + \mathrm{I\,J'} + \mathrm{L\,K'} + \mathrm{K\,L'}$	1
11−1	$\begin{matrix}11\\01\end{matrix}$	$\begin{matrix}10\\00\end{matrix}$	$= \mathrm{J\,I'} + \mathrm{I\,J'} + \mathrm{L\,K'} + \mathrm{K\,L'}$	2
15−1	$\begin{matrix}11\\11\end{matrix}$	$\begin{matrix}10\\00\end{matrix}$	$= \mathrm{J\,I'} + \mathrm{I\,J'} + \mathrm{L\,K'} + \mathrm{K\,L'}$	3
3−5	$\vartheta^{11}_{00}\,u+u'\,.$	$\vartheta^{10}_{10}\,u-u'$	$= \mathrm{J\,I'} - \mathrm{I\,J'} + \mathrm{L\,K'} - \mathrm{K\,L'}$	1
7−5	$\begin{matrix}11\\10\end{matrix}$	$\begin{matrix}10\\10\end{matrix}$	$= -\mathrm{J\,I'} + \mathrm{I\,J'} - \mathrm{L\,K'} + \mathrm{K\,L'}$	0
11−5	$\begin{matrix}11\\01\end{matrix}$	$\begin{matrix}10\\10\end{matrix}$	$= \mathrm{J\,I'} - \mathrm{I\,J'} + \mathrm{L\,K'} - \mathrm{K\,L'}$	3
15−5	$\begin{matrix}11\\11\end{matrix}$	$\begin{matrix}10\\10\end{matrix}$	$= -\mathrm{J\,I'} + \mathrm{I\,J'} - \mathrm{L\,K'} + \mathrm{K\,L'}$	2
3−9	$\vartheta^{11}_{00}\,u+u'\,.$	$\vartheta^{10}_{01}\,u-u'$	$= -i\mathrm{J\,I'} - i\mathrm{I\,J'} + i\mathrm{L\,K'} + i\mathrm{K\,L'}$	2
7−9	$\begin{matrix}11\\10\end{matrix}$	$\begin{matrix}10\\01\end{matrix}$	$= -i\mathrm{J\,I'} - i\mathrm{I\,J'} + i\mathrm{L\,K'} + i\mathrm{K\,L'}$	3
11−9	$\begin{matrix}11\\01\end{matrix}$	$\begin{matrix}10\\01\end{matrix}$	$= i\mathrm{J\,I'} + i\mathrm{I\,J'} - i\mathrm{L\,K'} - i\mathrm{K\,L'}$	0
15−9	$\begin{matrix}11\\11\end{matrix}$	$\begin{matrix}10\\01\end{matrix}$	$= i\mathrm{J\,I'} + i\mathrm{I\,J'} - i\mathrm{L\,K'} - i\mathrm{K\,L'}$	1
3−13	$\vartheta^{11}_{00}\,u+u'\,.$	$\vartheta^{10}_{11}\,u-u'$	$= -i\mathrm{J\,I'} + i\mathrm{I\,J'} + i\mathrm{L\,K'} - i\mathrm{K\,L'}$	3
7−13	$\begin{matrix}11\\10\end{matrix}$	$\begin{matrix}10\\11\end{matrix}$	$= i\mathrm{J\,I'} - i\mathrm{I\,J'} - i\mathrm{L\,K'} + i\mathrm{K\,L'}$	2
11−13	$\begin{matrix}11\\01\end{matrix}$	$\begin{matrix}10\\11\end{matrix}$	$= i\mathrm{J\,I'} - i\mathrm{I\,J'} - i\mathrm{L\,K'} + i\mathrm{K\,L'}$	1
15−13	$\begin{matrix}11\\11\end{matrix}$	$\begin{matrix}10\\11\end{matrix}$	$= -i\mathrm{J\,I'} + i\mathrm{I\,J'} + i\mathrm{L\,K'} - i\mathrm{K\,L'}$	0

Third set, 64 equations (continued).

								Suffixes.
0—2	$\vartheta^{00}_{00}\, u+u'\,.$	$\vartheta^{01}_{00}\, u-u'$	$=$	$\mathrm{I\,K'}$	$+\,\mathrm{J\,L'}$	$+\,\mathrm{K\,I'}$	$+\,\mathrm{L\,J'}$	0
4—2	$\begin{smallmatrix}00\\10\end{smallmatrix}$	$\begin{smallmatrix}01\\00\end{smallmatrix}$	$=$	$\mathrm{I\,K'}$	$+\,\mathrm{J\,L'}$	$+\,\mathrm{K\,I'}$	$+\,\mathrm{L\,J'}$	1
8—2	$\begin{smallmatrix}00\\01\end{smallmatrix}$	$\begin{smallmatrix}01\\00\end{smallmatrix}$	$=$	$\mathrm{I\,K'}$	$+\,\mathrm{J\,L'}$	$+\,\mathrm{K\,I'}$	$+\,\mathrm{L\,J'}$	2
12—2	$\begin{smallmatrix}00\\11\end{smallmatrix}$	$\begin{smallmatrix}01\\00\end{smallmatrix}$	$=$	$\mathrm{I\,K'}$	$+\,\mathrm{J\,L'}$	$+\,\mathrm{K\,I'}$	$+\,\mathrm{L\,J'}$	3
0—6	$\vartheta^{00}_{00}\, u+u'\,.$	$\vartheta^{01}_{10}\, u-u'$	$=$	$\mathrm{I\,K'}$	$-\,\mathrm{J\,L'}$	$+\,\mathrm{K\,I'}$	$-\,\mathrm{L\,J'}$	1
4—6	$\begin{smallmatrix}00\\10\end{smallmatrix}$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$=$	$\mathrm{I\,K'}$	$-\,\mathrm{J\,L'}$	$+\,\mathrm{K\,I'}$	$-\,\mathrm{L\,J'}$	0
8—6	$\begin{smallmatrix}00\\01\end{smallmatrix}$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$=$	$\mathrm{I\,K'}$	$-\,\mathrm{J\,L'}$	$+\,\mathrm{K\,I'}$	$-\,\mathrm{L\,J'}$	3
12—6	$\begin{smallmatrix}00\\11\end{smallmatrix}$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$=$	$\mathrm{I\,K'}$	$-\,\mathrm{J\,L'}$	$+\,\mathrm{K\,I'}$	$-\,\mathrm{L\,J'}$	2
0—10	$\vartheta^{00}_{00}\, u+u'\,.$	$\vartheta^{01}_{01}\, u-u'$	$=$	$i\mathrm{I\,K'}$	$+i\mathrm{J\,L'}$	$-i\mathrm{K\,I'}$	$-i\mathrm{L\,J'}$	2
4—10	$\begin{smallmatrix}00\\10\end{smallmatrix}$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$=$	$i\mathrm{I\,K'}$	$+i\mathrm{J\,L'}$	$-i\mathrm{K\,I'}$	$-i\mathrm{L\,J'}$	3
8—10	$\begin{smallmatrix}00\\01\end{smallmatrix}$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$=$	$i\mathrm{I\,K'}$	$+i\mathrm{J\,L'}$	$-i\mathrm{K\,I'}$	$-i\mathrm{L\,J'}$	0
12—10	$\begin{smallmatrix}00\\11\end{smallmatrix}$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$=$	$i\mathrm{I\,K'}$	$+i\mathrm{J\,L'}$	$-i\mathrm{K\,I'}$	$-i\mathrm{L\,J'}$	1
0—14	$\vartheta^{00}_{00}\, u+u'\,.$	$\vartheta^{01}_{11}\, u-u'$	$=$	$i\mathrm{I\,K'}$	$-i\mathrm{J\,L'}$	$-i\mathrm{K\,I'}$	$+i\mathrm{L\,J'}$	3
4—14	$\begin{smallmatrix}00\\10\end{smallmatrix}$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$=$	$i\mathrm{I\,K'}$	$-i\mathrm{J\,L'}$	$-i\mathrm{K\,I'}$	$+i\mathrm{L\,J'}$	2
8—14	$\begin{smallmatrix}00\\01\end{smallmatrix}$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$=$	$i\mathrm{I\,K'}$	$-i\mathrm{J\,L'}$	$-i\mathrm{K\,I'}$	$+i\mathrm{L\,J'}$	0
12—14	$\begin{smallmatrix}00\\11\end{smallmatrix}$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$=$	$i\mathrm{I\,K'}$	$-i\mathrm{J\,L'}$	$-i\mathrm{K\,I'}$	$+i\mathrm{L\,J'}$	1

THIRD set, 64 equations (concluded).

								Suffixes.
1−3	$\vartheta^{10}_{00}\, u+u'$.	$\vartheta^{11}_{00}\, u-u'$	$=$	$J\,K'$	$+\,I\,L'$	$+\,L\,I'$	$+\,K\,J'$	0
5−3	$\begin{matrix}10\\10\end{matrix}$	$\begin{matrix}11\\00\end{matrix}$	$=$	$J\,K'$	$+\,I\,L'$	$+\,L\,I'$	$+\,K\,J'$	1
9−3	$\begin{matrix}10\\01\end{matrix}$	$\begin{matrix}11\\00\end{matrix}$	$=$	$J\,K'$	$+\,I\,L'$	$+\,L\,I'$	$+\,K\,J'$	2
13−3	$\begin{matrix}10\\11\end{matrix}$	$\begin{matrix}11\\00\end{matrix}$	$=$	$J\,K'$	$+\,I\,L'$	$+\,L\,I'$	$+\,K\,J'$	3
1−7	$\vartheta^{10}_{00}\, u+u'$.	$\vartheta^{11}_{10}\, u-u'$	$=$	$J\,K'$	$-\,I\,L'$	$+\,L\,I'$	$-\,K\,J'$	1
5−7	$\begin{matrix}10\\10\end{matrix}$	$\begin{matrix}11\\10\end{matrix}$	$=-$	$J\,K'$	$+\,I\,L'$	$-\,L\,I'$	$+\,K\,J'$	0
9−7	$\begin{matrix}10\\01\end{matrix}$	$\begin{matrix}11\\10\end{matrix}$	$=$	$J\,K'$	$-\,I\,L'$	$+\,L\,I'$	$-\,K\,J'$	3
13−7	$\begin{matrix}10\\11\end{matrix}$	$\begin{matrix}11\\10\end{matrix}$	$=-$	$J\,K'$	$+\,I\,L'$	$-\,L\,I'$	$+\,K\,J'$	2
1−11	$\vartheta^{10}_{00}\, u+u'$.	$\vartheta^{11}_{01}\, u-u'$	$=$	$iJ\,K'$	$+iI\,L'$	$-iL\,I'$	$-iK\,J'$	2
5−11	$\begin{matrix}10\\10\end{matrix}$	$\begin{matrix}11\\01\end{matrix}$	$=$	$iJ\,K'$	$+iI\,L'$	$-iL\,I'$	$-iK\,J'$	3
9−11	$\begin{matrix}10\\01\end{matrix}$	$\begin{matrix}11\\01\end{matrix}$	$=$	$iJ\,K'$	$+iI\,L'$	$-iL\,I'$	$-iK\,J'$	0
13−11	$\begin{matrix}10\\11\end{matrix}$	$\begin{matrix}11\\01\end{matrix}$	$=$	$iJ\,K'$	$+iI\,L'$	$-iL\,I'$	$-iK\,J'$	1
1−15	$\vartheta^{10}_{00}\, u+u'$.	$\vartheta^{11}_{11}\, u-u'$	$=$	$iJ\,K'$	$-iI\,L'$	$-iL\,I'$	$+iK\,J'$	3
5−15	$\begin{matrix}10\\10\end{matrix}$	$\begin{matrix}11\\11\end{matrix}$	$=-$	$iJ\,K'$	$+iI\,L'$	$+iL\,I'$	$-iK\,J'$	2
9−15	$\begin{matrix}10\\01\end{matrix}$	$\begin{matrix}11\\11\end{matrix}$	$=$	$iJ\,K'$	$-iI\,L'$	$-iL\,I'$	$+iK\,J'$	1
13−15	$\begin{matrix}10\\11\end{matrix}$	$\begin{matrix}11\\11\end{matrix}$	$=-$	$iJ\,K'$	$+iI\,L'$	$+iL\,I'$	$-iK\,J'$	0

89. Fourth set, 64 equations.

				Suffixes.
3−0	$\vartheta^{11}_{00}\, u+u' \,.$	$\vartheta^{00}_{00}\, u-u'$	$= \mathrm{M\,M'} + \mathrm{N\,N'} + \mathrm{P\,P'} + \mathrm{Q\,Q'}$	0
7−0	$\begin{matrix}11\\10\end{matrix}$	$\begin{matrix}00\\00\end{matrix}$	$= \mathrm{M\,M'} + \mathrm{N\,N'} + \mathrm{P\,P'} + \mathrm{Q\,Q'}$	1
11−0	$\begin{matrix}11\\01\end{matrix}$	$\begin{matrix}00\\00\end{matrix}$	$= \mathrm{M\,M'} + \mathrm{N\,N'} + \mathrm{P\,P'} + \mathrm{Q\,Q'}$	2
15−0	$\begin{matrix}11\\11\end{matrix}$	$\begin{matrix}00\\00\end{matrix}$	$= \mathrm{M\,M'} + \mathrm{N\,N'} + \mathrm{P\,P'} + \mathrm{Q\,Q'}$	3
3−4	$\vartheta^{11}_{00}\, u+u' \,.$	$\vartheta^{00}_{10}\, u-u'$	$= -i\mathrm{M\,M'} + i\mathrm{N\,N'} - i\mathrm{P\,P'} + i\mathrm{Q\,Q'}$	1
7−4	$\begin{matrix}11\\10\end{matrix}$	$\begin{matrix}00\\10\end{matrix}$	$= +i\mathrm{M\,M'} - i\mathrm{N\,N'} + i\mathrm{P\,P'} - i\mathrm{Q\,Q'}$	0
11−4	$\begin{matrix}11\\01\end{matrix}$	$\begin{matrix}00\\10\end{matrix}$	$= -i\mathrm{M\,M'} + i\mathrm{N\,N'} - i\mathrm{P\,P'} + i\mathrm{Q\,Q'}$	3
15−4	$\begin{matrix}11\\11\end{matrix}$	$\begin{matrix}00\\10\end{matrix}$	$= +i\mathrm{M\,M'} - i\mathrm{N\,N'} + i\mathrm{P\,P'} - i\mathrm{Q\,Q'}$	2
3−8	$\vartheta^{11}_{00}\, u+u' \,.$	$\vartheta^{00}_{01}\, u-u'$	$= -i\mathrm{M\,M'} - i\mathrm{N\,N'} + i\mathrm{P\,P'} + i\mathrm{Q\,Q'}$	2
7−8	$\begin{matrix}11\\10\end{matrix}$	$\begin{matrix}00\\01\end{matrix}$	$= -i\mathrm{M\,M'} - i\mathrm{N\,N'} + i\mathrm{P\,P'} + i\mathrm{Q\,Q'}$	3
11−8	$\begin{matrix}11\\01\end{matrix}$	$\begin{matrix}00\\01\end{matrix}$	$= i\mathrm{M\,M'} + i\mathrm{N\,N'} - i\mathrm{P\,P'} - i\mathrm{Q\,Q'}$	0
15−8	$\begin{matrix}11\\11\end{matrix}$	$\begin{matrix}00\\01\end{matrix}$	$= i\mathrm{M\,M'} + i\mathrm{N\,N'} - i\mathrm{P\,P'} - i\mathrm{Q\,Q'}$	1
3−12	$\vartheta^{11}_{00}\, u+u' \,.$	$\vartheta^{00}_{11}\, u-u'$	$= -\mathrm{M\,M'} + \mathrm{N\,N'} + \mathrm{P\,P'} - \mathrm{Q\,Q'}$	3
7−12	$\begin{matrix}11\\10\end{matrix}$	$\begin{matrix}00\\11\end{matrix}$	$= +\mathrm{M\,M'} - \mathrm{N\,N'} - \mathrm{P\,P'} + \mathrm{Q\,Q'}$	2
11−12	$\begin{matrix}11\\01\end{matrix}$	$\begin{matrix}00\\11\end{matrix}$	$= +\mathrm{M\,M'} - \mathrm{N\,N'} - \mathrm{P\,P'} + \mathrm{Q\,Q'}$	1
15−12	$\begin{matrix}11\\11\end{matrix}$	$\begin{matrix}00\\11\end{matrix}$	$= -\mathrm{M\,M'} + \mathrm{N\,N'} + \mathrm{P\,P'} - \mathrm{Q\,Q'}$	0

FOURTH set, 64 equations (continued).

				Suffixes.
2−1	$\vartheta^{01}_{00}\,u+u'\,.$	$\vartheta^{10}_{00}\,u-u'$	$= MN' + NM' + PQ' + QP'$	0
6−1	$\genfrac{}{}{0pt}{}{01}{10}$	$\genfrac{}{}{0pt}{}{10}{00}$	$= MN' + NM' + PQ' + QP'$	1
10−1	$\genfrac{}{}{0pt}{}{01}{01}$	$\genfrac{}{}{0pt}{}{10}{00}$	$= MN' + NM' + PQ' + QP'$	2
14−1	$\genfrac{}{}{0pt}{}{01}{11}$	$\genfrac{}{}{0pt}{}{10}{00}$	$= MN' + NM' + PQ' + QP'$	3
2−5	$\vartheta^{01}_{00}\,u+u'\,.$	$\vartheta^{10}_{10}\,u-u'$	$= iMN' - iNM' + iPQ' - iQP'$	1
6−5	$\genfrac{}{}{0pt}{}{01}{10}$	$\genfrac{}{}{0pt}{}{10}{10}$	$= iMN' - iNM' + iPQ' - iQP'$	0
10−5	$\genfrac{}{}{0pt}{}{01}{01}$	$\genfrac{}{}{0pt}{}{10}{10}$	$= iMN' - iNM' + iPQ' - iQP'$	3
14−5	$\genfrac{}{}{0pt}{}{01}{11}$	$\genfrac{}{}{0pt}{}{10}{10}$	$= iMN' - iNM' + iPQ' - iQP'$	2
2−9	$\vartheta^{01}_{00}\,u+u'\,.$	$\vartheta^{10}_{01}\,u-u'$	$= -iMN' - iNM' + iPQ' + iQP'$	2
6−9	$\genfrac{}{}{0pt}{}{01}{10}$	$\genfrac{}{}{0pt}{}{10}{01}$	$= -iMN' - iNM' + iPQ' + iQP'$	3
10−9	$\genfrac{}{}{0pt}{}{01}{01}$	$\genfrac{}{}{0pt}{}{10}{01}$	$= iMN' + iNM' - iPQ' - iQP'$	0
14−9	$\genfrac{}{}{0pt}{}{01}{11}$	$\genfrac{}{}{0pt}{}{10}{01}$	$= iMN' + iNM' - iPQ' - iQP'$	1
2−13	$\vartheta^{01}_{00}\,u+u'\,.$	$\vartheta^{10}_{11}\,u-u'$	$= MN' - NM' - PQ' + QP'$	3
6−13	$\genfrac{}{}{0pt}{}{01}{11}$	$\genfrac{}{}{0pt}{}{10}{11}$	$= MN' - NM' - PQ' + QP'$	2
10−13	$\genfrac{}{}{0pt}{}{01}{01}$	$\genfrac{}{}{0pt}{}{10}{11}$	$= -MN' + NM' + PQ' - QP'$	1
14−13	$\genfrac{}{}{0pt}{}{01}{11}$	$\genfrac{}{}{0pt}{}{10}{11}$	$= -MN' + NM' + PQ' - QP'$	0

FOURTH set, 64 equations (continued).

				Suffixes.
1−2	$\vartheta^{10}_{00}\,u+u'$.	$\vartheta^{01}_{00}\,u-u'$	$= \ \ MP' + NQ' + PM' + QN'$	0
5−2	$\genfrac{}{}{0pt}{}{10}{10}$	$\genfrac{}{}{0pt}{}{01}{00}$	$= \ \ MP' + NQ' + PM' + QN'$	1
9−2	$\genfrac{}{}{0pt}{}{10}{01}$	$\genfrac{}{}{0pt}{}{01}{00}$	$= \ \ MP' + NQ' + PM' + QN'$	2
13−2	$\genfrac{}{}{0pt}{}{10}{11}$	$\genfrac{}{}{0pt}{}{01}{00}$	$= \ \ MP' + NQ' + PM' + QN'$	3
1−6	$\vartheta^{10}_{00}\,u+u'$.	$\vartheta^{01}_{10}\,u-u'$	$= -iMP' + iNQ' - iPM' + iQN'$	1
5−6	$\genfrac{}{}{0pt}{}{10}{10}$	$\genfrac{}{}{0pt}{}{01}{10}$ *	$= \ \ iMP' - iNQ' + iPM' - iQN'$	0
9−6	$\genfrac{}{}{0pt}{}{10}{01}$	$\genfrac{}{}{0pt}{}{01}{10}$	$= -iMP' + iNQ' - iPM' + iQN'$	3
13−6	$\genfrac{}{}{0pt}{}{10}{11}$	$\genfrac{}{}{0pt}{}{01}{10}$	$= \ \ iMP' - iNQ' + iPM' - iQN'$	2
1−10	$\vartheta^{10}_{00}\,u+u'$.	$\vartheta^{01}_{01}\,u-u'$	$= \ \ iMP' + iNQ' - iPM' - iQN'$	2
5−10	$\genfrac{}{}{0pt}{}{10}{10}$	$\genfrac{}{}{0pt}{}{01}{01}$	$= \ \ iMP' + iNQ' - iPM' - iQN'$	3
9−10	$\genfrac{}{}{0pt}{}{10}{01}$	$\genfrac{}{}{0pt}{}{01}{01}$	$= \ \ iMP' + iNQ' - iPM' - iQN'$	0
13−10	$\genfrac{}{}{0pt}{}{10}{11}$	$\genfrac{}{}{0pt}{}{01}{01}$	$= \ \ iMP' + iNQ' - iPM' - iQN'$	1
1−14	$\vartheta^{10}_{00}\,u+u'$.	$\vartheta^{01}_{11}\,u-u'$	$= \ \ MP' - NQ' - PM' + QN'$	3
5−14	$\genfrac{}{}{0pt}{}{10}{10}$	$\genfrac{}{}{0pt}{}{01}{11}$	$= -MP' + NQ' + PM' - QN'$	2
9−14	$\genfrac{}{}{0pt}{}{10}{01}$	$\genfrac{}{}{0pt}{}{01}{11}$	$= \ \ MP' - NQ' - PM' + QN'$	1
13−14	$\genfrac{}{}{0pt}{}{10}{11}$	$\genfrac{}{}{0pt}{}{01}{11}$	$= -MP' + NQ' + PM' - QN'$	0

FOURTH set, 64 equations (concluded).

				Suffixes.
0—3	$\vartheta^{00}_{00}\,u+u'\,.$	$\vartheta^{11}_{00}\,u-u'$	$= \mathrm{M\,Q'} + \mathrm{N\,P'} + \mathrm{P\,N'} + \mathrm{Q\,M'}$	0
4—3	${}^{00}_{10}$	${}^{11}_{00}$	$= \mathrm{M\,Q'} + \mathrm{N\,P'} + \mathrm{P\,N'} + \mathrm{Q\,M'}$	1
8—3	${}^{00}_{01}$	${}^{11}_{00}$	$= \mathrm{M\,Q'} + \mathrm{N\,P'} + \mathrm{P\,N'} + \mathrm{Q\,M'}$	2
12—3	${}^{00}_{11}$	${}^{11}_{00}$	$= \mathrm{M\,Q'} + \mathrm{N\,P'} + \mathrm{P\,N'} + \mathrm{Q\,M'}$	3
0—7	$\vartheta^{00}_{00}\,u+u'\,.$	$\vartheta^{11}_{10}\,u-u'$	$= i\mathrm{M\,Q'} - i\mathrm{N\,P'} + i\mathrm{P\,N'} - i\mathrm{Q\,M'}$	1
4—7	${}^{00}_{10}$	${}^{11}_{10}$	$= i\mathrm{M\,Q'} - i\mathrm{N\,P'} + i\mathrm{P\,N'} - i\mathrm{Q\,M'}$	0
8—7	${}^{00}_{01}$	${}^{11}_{10}$	$= i\mathrm{M\,Q'} - i\mathrm{N\,P'} + i\mathrm{P\,N'} - i\mathrm{Q\,M'}$	3
12—7	${}^{00}_{11}$	${}^{11}_{10}$	$= i\mathrm{M\,Q'} - i\mathrm{N\,P'} + i\mathrm{P\,N'} - i\mathrm{Q\,M'}$	2
0—11	$\vartheta^{00}_{00}\,u+u'\,.$	$\vartheta^{11}_{01}\,u-u'$	$= i\mathrm{M\,Q'} + i\mathrm{N\,P'} - i\mathrm{P\,N'} - i\mathrm{Q\,M'}$	2
4—11	${}^{00}_{10}$	${}^{11}_{01}$	$= i\mathrm{M\,Q'} + i\mathrm{N\,P'} - i\mathrm{P\,N'} - i\mathrm{Q\,M'}$	3
8—11	${}^{00}_{01}$	${}^{11}_{01}$	$= i\mathrm{M\,Q'} + i\mathrm{N\,P'} - i\mathrm{P\,N'} - i\mathrm{Q\,M'}$	0
12—11	${}^{00}_{11}$	${}^{11}_{01}$	$= i\mathrm{M\,Q'} + i\mathrm{N\,P'} - i\mathrm{P\,N'} - i\mathrm{Q\,M'}$	1
0—15	$\vartheta^{00}_{00}\,u+u'\,.$	$\vartheta^{11}_{11}\,u-u'$	$= - \mathrm{M\,Q'} + \mathrm{N\,P'} + \mathrm{P\,N'} - \mathrm{Q\,M'}$	3
4—15	${}^{00}_{10}$	${}^{11}_{11}$	$= - \mathrm{M\,Q'} + \mathrm{N\,P'} + \mathrm{P\,N'} - \mathrm{Q\,M'}$	2
8—15	${}^{00}_{01}$	${}^{11}_{11}$	$= - \mathrm{M\,Q'} + \mathrm{N\,P'} + \mathrm{P\,N'} - \mathrm{Q\,M'}$	1
12—15	${}^{00}_{11}$	${}^{11}_{11}$	$= - \mathrm{M\,Q'} + \mathrm{N\,P'} + \mathrm{P\,N'} - \mathrm{Q\,M'}$	0

90. I re-arrange these in sets of 16 equations, the equations of the first or square-set of 16 being taken as they stand, but those of the other sets being combined in pairs by addition and subtraction as will be seen. And I now drop altogether the characteristics, retaining only the current numbers: thus in the set of equations next written down, the first equation is

$$\vartheta_0(u+u').\vartheta_0(u-u')=XX'+YY'+ZZ'+WW':$$

in the second set, the first equation is

$$\tfrac{1}{2}\{\vartheta_4(u+u').\vartheta_0(u-u')+\vartheta_0(u+u')\vartheta_4(u-u')\}=X_1X_1'+Z_1Z_1',$$

and so in other cases.

FIRST or square-set of 16.

$\overset{u+u'}{\vartheta}$	$\overset{u-u'}{\vartheta}$	(Suffixes 0.)
0	0	$= \;\; X\,X' + Y\,Y' + Z\,Z' + W\,W'$
4	4	$= \;\; X\,X' - Y\,Y' + Z\,Z' - W\,W'$
8	8	$= \;\; X\,X' + Y\,Y' - Z\,Z' - W\,W'$
12	12	$= \;\; X\,X' - Y\,Y' - Z\,Z' + W\,W'$
1	1	$= \;\; Y\,X' + X\,Y' + W\,Z' + Z\,W'$
5	5	$= -Y\,X' + X\,Y' - W\,Z' + Z\,W'$
9	9	$= \;\; Y\,X' + X\,Y' - W\,Z' - Z\,W'$
13	13	$= -Y\,X' + X\,Y' + W\,Z' - Z\,W'$
2	2	$= \;\; Z\,X' + W\,Y' + X\,Z' + Y\,W'$
6	6	$= \;\; Z\,X' - W\,Y' + X\,Z' - Y\,W'$
10	10	$= -Z\,X' - W\,Y' + X\,Z' + Y\,W'$
14	14	$= -Z\,X' + W\,Y' + X\,Z' - Y\,W'$
3	3	$= \;\; W\,X' + Z\,Y' + Y\,Z' + X\,W'$
7	7	$= -W\,X' + Z\,Y' - Y\,Z' + X\,W'$
11	11	$= -W\,X' - Z\,Y' + Y\,Z' + X\,W'$
15	15	$= \;\; W\,X' - Z\,Y' - Y\,Z' + X\,W'$

91. SECOND set of 16.

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}$	$.\,\overset{u-u'}{\vartheta}$	$+\,\overset{u+u'}{\vartheta}$	$.\,\overset{u-u'}{\vartheta}\}$	(Suffixes 1.)
4	0	0	4	$= \;\; X\,X' + Z\,Z'$
12	8	8	12	$X\,X' - Z\,Z'$
5	1	1	5	$Y\,X' + W\,Z'$
13	9	9	13	$Y\,X' - W\,Z'$
6	2	2	6	$Z\,X' + X\,Z'$
14	10	10	14	$-Z\,X' + X\,Z'$
7	3	3	7	$W\,X' + Y\,Z'$
15	11	11	15	$-W\,X' + Y\,Z'$

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}$	$.\,\overset{u-u'}{\vartheta}$	$-\,\overset{u+u'}{\vartheta}$	$.\,\overset{u-u'}{\vartheta}\}$	(Suffixes 1.)
4	0	0	4	$= \;\; Y\,Y' + W\,W'$
12	8	8	12	$Y\,Y' - W\,W'$
5	1	1	5	$X\,Y' + Z\,W'$
13	9	9	13	$X\,Y' - Z\,W'$
6	2	2	6	$W\,Y' + Y\,W'$
14	10	10	14	$-W\,Y' + Y\,W'$
7	3	3	7	$Z\,Y' + X\,W'$
15	11	11	15	$-Z\,Y' + X\,W'$

92. Third set of 16.

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}+\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 2.)

8	0	0	8	=	$X\,X'+Y\,Y'$
12	4	4	12		$X\,X'-Y\,Y'$
9	1	1	9		$Y\,X'+X\,Y'$
13	5	5	13		$-Y\,X'+X\,Y'$
10	2	2	10		$Z\,X'+W\,Y'$
14	6	6	14		$Z\,X'-W\,Y'$
11	3	3	11		$W\,X'+Z\,Y'$
15	7	7	15		$-W\,X'+Z\,Y'$

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}-\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 2.)

8	0	0	8	=	$Z\,Z'+W\,W'$
12	4	4	12		$Z\,Z'-W\,W'$
9	1	1	9		$W\,Z'+Z\,W'$
13	5	5	13		$-W\,Z'+Z\,W'$
10	2	2	10		$X\,Z'+Y\,W'$
14	6	6	14		$X\,Z'-Y\,W'$
11	3	3	11		$Y\,Z'+X\,W'$
15	7	7	15		$-Y\,Z'+X\,W'$

93. Fourth set of 16.

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}+\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 3.)

12	0	0	12	=	$X\,X'+W\,W'$
8	4	4	8		$X\,X'-W\,W'$
13	1	1	13		$Y\,X'+Z\,W'$
9	5	5	9		$Y\,X'-Z\,W'$
14	2	2	14		$Z\,X'+Y\,W'$
10	6	6	10		$Z\,X'-Y\,W'$
15	3	3	15		$W\,X'+X\,W'$
11	7	7	11		$W\,X'-X\,W'$

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}-\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 3.)

12	0	0	12	=	$Y\,Y'+Z\,Z'$
8	4	4	8		$-Y\,Y'+Z\,Z'$
13	1	1	13		$X\,Y'+W\,Z'$
9	5	5	9		$-X\,Y'+W\,Z'$
14	2	2	14		$W\,Y'+X\,Z'$
10	6	6	10		$-W\,Y'+X\,Z'$
15	3	3	15		$Z\,Y'+Y\,Z'$
11	7	7	11		$-Z\,Y'+Y\,Z'$

94. FIFTH set of 16.

$$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}+\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$$ (Suffixes 0.)

1	0	0	1	$=$	$E+G$	$.\ E'+G'$	$+$	$F+H$	$.\ F'+H'$
5	4	4	5		$i.E-G$	,,	$+$	$i.F-H$	,,
9	8	8	9		$E+G$	,,	$-$	$.F+H$	,,
13	12	12	13		$i.E-G$	,,	$-$	$i.F-H$	,,
3	2	2	3		$F+H$	,,	$+$	$E+G$	,,
7	6	6	7		$i.F-H$	,,	$+$	$i.E-G$	,,
11	10	10	11		$-\ .F+H$	,,	$+$	$E+G$	,,
15	14	14	15		$-i.F-H$	,,	$+$	$i.E-G$	,,

$$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}-\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$$ (Suffixes 0.)

1	0	0	1	$=$	$E-G$	$.\ E'-G'$	$+$	$F-H$	$.\ F'-H'$
5	4	4	5		$i.E+G$	,,	$+$	$i.F+H$	,,
9	8	8	9		$E-G$	,,	$-$	$.F-H$	,,
13	12	12	13		$i.E+G$	,,	$-$	$i.F+H$	,,
3	2	2	3		$F-H$	,,	$+$	$E-G$	,,
7	6	6	7		$i.F+H$	,,	$+$	$i.E+G$	,,
11	10	10	11		$-\ .F-H$	,,	$+$	$E-G$	,,
15	14	14	15		$-i.F+H$	,,	$+$	$i.E+G$	,,

95. SIXTH set of 16.

$$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}+\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$$ (Suffixes 1.)

5	0	0	5	$=$	$E-iG$	$.\ E'+iG'$	$+$	$F-iH$	$.\ F'+iH'$
1	4	4	1		$-i.E+iG$	,,	$-$	$i.F+iH$	,,
8	13	13	8		$E-iG$	,,	$-$	$.F-iH$	,,
9	12	12	9		$-i.E+iG$	,,	$+$	$i.F+iH$	,,
7	2	2	7		$F-iH$	,,	$+$	$E-iG$	,,
3	6	6	3		$-i.F+iH$	,,	$-$	$i.E+iG$	,,
15	10	10	15		$-\ .F-iH$	,,	$+$	$E-iG$	,,
11	14	14	11		$i.F+iH$	,,	$-$	$i.E+iG$	,,

$$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}-\overset{u+u}{\vartheta}\,.\,\overset{u-u}{\vartheta}\}$$ (Suffixes 1.)

5	0	0	5	$=$	$E+iG$	$.\ E'-iG'$	$+$	$F+iH$	$.\ F'-iH'$
1	4	4	1		$-i.E-iG$	,,	$-$	$i.F-iH$	,,
13	8	8	13		$E+iG$	,,	$-$	$.F+iH$	,,
9	12	12	9		$-i.E-iG$	,,	$+$	$i.F-iH$	,,
7	2	2	7		$F+iH$	,,	$+$	$E+iG$	,,
3	6	6	3		$-i.F-iH$	,,	$-$	$i.E-iG$	,,
15	10	10	15		$-\ .F+iH$	,,	$+$	$.E+iG$	,,
11	14	14	11		$+i.F-iH$	,,	$-$	$i.E-iG$	,,

96. SEVENTH set of 16.

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}+\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 2.)

9	0	0	9	=	$E+G$	. $E'+G'$	+	$F-H$	. $F'-H'$
13	4	4	13		$i.E-G$	,,	+	$i.F+H$	,,
1	8	8	1		$E+G$	,,	−	$.F-H$	,,
5	12	12	5		$i.E-G$	,,	−	$i.F+H$	,,
11	2	2	11		$F+H$	,,	+	$E-G$	,,
15	6	6	15		$i.F-H$	,,	+	$i.E+G$	,,
3	10	10	3		$F+H$	,,	−	$.E-G$	,,
7	14	14	7		$i.F-H$	,,	−	$i.E+G$	,,

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}-\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 2.)

9	0	0	9	=	$E-G$	. $E'-G'$	+	$F+H$	. $F'+H'$
13	4	4	13		$i.E+G$	,,	+	$i.F-H$	,,
1	8	8	1		$E-G$	,,	−	$.F+H$	,,
5	12	12	5		$i.E+G$	,,	−	$i.F-H$	,,
11	2	2	11		$F-H$	,,	+	$E+G$	,,
15	6	6	15		$i.F+H$	,,	+	$i.E-G$	,,
3	10	10	3		$F-H$	,,	−	$.E+G$	,,
7	14	14	7		$i.F+H$	,,	−	$i.E-G$	,,

97. EIGHTH set of 16.

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}+\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 3.)

13	0	0	13	=	$E-iG$	. $E'+iG'$	+	$F+iH$	. $F'-iH'$
9	4	4	9		$-i.E+iG$	,,	−	$i.F-iH$	,,
5	8	8	5		$E-iG$	,,	−	$.F+iH$	,,
1	12	12	1		$-i.E+iG$	,,	+	$i.F-iH$	,,
15	2	2	15		$F-iH$	,,	+	$E+iG$	,,
11	6	6	11		$-i.F+iH$	,,	−	$i.E-iG$	,,
7	10	10	7		$F-iH$	,,	−	$.E+iG$	,,
3	14	14	3		$-i.F+iH$	,,	+	$i.E-iG$	,,

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}-\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 3.)

13	0	0	13	=	$E+iG$	. $E'-iG'$	+	$F-iH$	. $F'+iH'$
9	4	4	9		$-i.E-iG$	,,	−	$i.F+iH$	,,
5	8	8	5		$E+iG$	,,	−	$.F-iH$	,,
1	12	12	1		$-i.E-iG$	,,	+	$i.F+iH$	,,
15	2	2	15		$F+iH$	,,	+	$E-iG$	,,
11	6	6	11		$-i.F-iH$	,,	−	$i.E+iG$	,,
7	10	10	7		$F+iH$	,,	−	$.E-iG$	,,
3	14	14	3		$-i.F-iH$	,,	+	$i.E+iG$	,,

98. NINTH set of 16.

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}+\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 0.)

2	0	0	2	=	I+K	. I'+K'	+	J+L	. J'+L'
6	4	4	6		I+K	„	−	.J+L	„
10	8	8	10		i.I−K	„	+	i.J−L	„
14	12	12	14		i.I−K	„	−	i.J−L	„
3	1	1	3		J+L	„	+	I+K	„
7	5	5	7		− .J+L	„	−	.I+K	„
11	9	9	11		i.J−L	„	+	i.I−K	„
15	13	13	15		−i.J−L	„	−	i.I−K	„

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}-\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 0.)

2	0	0	2	=	I−K	. I'−K'	+	J−L	. J'−L'
6	4	4	6		I−K	„	−	.J−L	„
10	8	8	10		i.I+K	„	+	i.J+L	„
14	12	12	14		i.I+K	„	−	i.J+L	„
3	1	1	3		J−L	„	+	I−K	„
7	5	5	7		− .J−L	„	+	I−K	„
11	9	9	11		i.J+L	„	+	i.I+K	„
15	13	13	15		−i.J+L	„	+	i.I+K	„

99. TENTH set of 16.

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}+\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 1.)

6	0	0	6	=	I+K	. I'+K'	+	J−L	. J'−L'
2	4	4	2		I+K	„	−	.J−L	„
14	8	8	14		i.I−K	„	+	i.J+L	„
10	12	12	10		i.I−K	„	−	i.J+L	„
7	1	1	7		J+L	„	+	I−K	„
3	5	5	3		J+L	„	−	.I−K	„
15	9	9	15		i.J−L	„	+	i.I+K	„
11	13	13	11		i.J−L	„	−	i.I+K	„

$\tfrac{1}{2}\{\overset{u+u}{\vartheta}\,.\,\overset{u-u'}{\vartheta}-\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 1.)

6	0	0	6	=	I−K	. I'−K'	+	J+L	. J'+L'
2	4	4	2		I−K	„	−	.J+L	„
14	8	8	14		i.I+K	„	+	i.J−L	„
10	12	12	10		i.I+K	„	−	i.J−L	„
7	1	1	7		J−L	„	+	I+K	„
3	5	5	3		J−L	„	−	.I+K	„
15	9	9	15		i.J+L	„	+	i.I−K	„
11	13	13	11		i.J+L	„	−	i.I−K	„

100. Eleventh set of 16.

$$\tfrac{1}{2}\{\overset{u+u'}{\vartheta} . \overset{u-u'}{\vartheta} + \overset{u+u'}{\vartheta} . \overset{u-u'}{\vartheta}\}$$ (Suffixes 2.)

10	0	0	10	=	$I-iK$	$. I'+iK'$	$+$	$.J-iL$	$. J'+iL'$
14	4	4	14		$I-iK$	,,	$-$	$.J-iL$	,,
2	8	8	2		$-i.I+iK$	,,	$-$	$i.J+iL$	,,
6	12	12	6		$-i.I+iK$	,,	$+$	$i.J+iL$	,,
11	1	1	11		$J-iL$	,,	$+$	$I-iK$	,,
15	5	5	15		$-\ .J-iL$	,,	$+$	$I-iK$	,,
3	9	9	3		$-i.J+iL$	,,	$-$	$i.I+iK$	,,
7	13	13	7		$+i.J+iL$	,,	$-$	$i.I+iK$	,,

$$\tfrac{1}{2}\{\overset{u+u'}{\vartheta} . \overset{u-u'}{\vartheta} - \overset{u+u'}{\vartheta} . \overset{u-u}{\vartheta}\}$$ (Suffixes 2.)

10	0	0	10	=	$I+iK$	$. I'-iK'$	$+$	$J+iL$	$. J'-iL'$
14	4	4	14		$I+iK$	,,	$-$	$J+iL$	,,
2	8	8	2		$-i.I-iK$	,,	$-$	$i.J-iL$	,,
6	12	12	6		$-i.I-iK$	,,	$+$	$i.J-iL$	,,
11	1	1	11		$J+iL$	,,	$+$	$I+iK$	,,
15	5	5	15		$-\ .J+iL$	,,	$+$	$I+iK$	,,
3	9	9	3		$-i.J-iL$	,,	$-$	$i.I-iK$	,,
7	13	13	7		$+i.J-iL$	,,	$-$	$i.I-iK$	,,

101. Twelfth set of 16.

$$\tfrac{1}{2}\{\overset{u+u}{\vartheta} . \overset{u-u}{\vartheta} + \overset{u+u'}{\vartheta} . \overset{u-u'}{\vartheta}\}$$ (Suffixes 3.)

14	0	0	14	=	$I-iK$	$. I'+iK'$	$+$	$J+iL$	$. J'-iL'$
10	4	4	10		$I-iK$	,,	$-$	$.J+iL$	,,
6	8	8	6		$-i.I+iK$	,,	$-$	$i.J-iL$	,,
2	12	12	2		$-i.I+iK$	,,	$+$	$i.J-iL$	,,
15	1	1	15		$J-iL$	,,	$+$	$I+iK$	,,
11	5	5	11		$J-iL$	,,	$-$	$.I+iK$	,,
7	9	9	7		$-i.J+iL$	,,	$-$	$i.I-iK$	,,
3	13	13	3		$-i.J+iL$	,,	$+$	$i.I-iK$	,,

$$\tfrac{1}{2}\{\overset{u+u'}{\vartheta} . \overset{u-u}{\vartheta} - \overset{u+u'}{\vartheta} . \overset{u-u'}{\vartheta}\}$$ (Suffixes 3.)

14	0	0	14	=	$I+iK$	$. I'-iK'$	$+$	$J-iL$	$. J'+iL'$
10	4	4	10		$I+iK$	,,	$-$	$.J-iL$	,,
6	8	8	6		$-i.I-iK$	,,	$-$	$i.J+iL$	,,
2	12	12	2		$-i.I-iK$	,,	$+$	$i.J+iL$	,,
15	1	1	15		$J+iL$	,,	$+$	$I-iK$	,,
11	5	5	11		$J+iL$	,,	$-$	$.I-iK$	,,
7	9	9	7		$-i.J-iL$	,,	$-$	$i.I+iK$	,,
3	13	13	3		$-i.J-iL$	,,	$+$	$i.I+iK$	,,

102. THIRTEENTH set of 16.

$\frac{1}{2}\{\overset{u+u}{\vartheta}\ .\ \overset{u-u'}{\vartheta}+\overset{u+u}{\vartheta}\ .\ \overset{u-u'}{\vartheta}\}$ (Suffixes 0.)

3	0	0	3	=	$M+Q$	.	$M'+Q'$	+	$N+P$	. $N'+P'$
7	4	4	7		$i.M-Q$		,,	−	$i.N-P$	,,
11	8	8	11		$i.M-Q$		,,	+	$i.N-P$	,,
15	12	12	15		$-\ .M+Q$		,,	+	$N+P$	,,
2	1	1	2		$N+P$		,,	+	$M+Q$	,,
6	5	5	6		$-i.N-P$		,,	+	$i.M-Q$	,,
10	9	9	10		$i.N-P$		,,	+	$i.M-Q$	,,
14	13	13	14		$N+P$		,,	−	$.M+Q$	,,

$\frac{1}{2}\{\overset{u+u}{\vartheta}\ .\ \overset{u-u'}{\vartheta}-\overset{u+u'}{\vartheta}\ .\ \overset{u-u}{\vartheta}\}$ (Suffixes 0.)

3	0	0	3	=	$M-Q$	.	$M'-Q'$	+	$N-P$	. $N'-P'$
7	4	4	7		$i.M+Q$		,,	−	$i.N+P$	,,
11	8	8	11		$i.M+Q$		,,	+	$i.N+P$	,,
15	12	12	15		$-\ .M-Q$		,,	+	$N-P$	,,
2	1	1	2		$N-P$		,,	+	$M-Q$	,,
6	5	5	6		$-i.N+P$		,,	+	$i.M+Q$	,,
10	9	9	10		$i.N+P$		,,	+	$i.M+Q$	,,
14	13	13	14		$N-P$		,,	−	$.M-Q$	,,

103. FOURTEENTH set of 16.

$\frac{1}{2}\{\overset{u+u'}{\vartheta}\ .\ \overset{u-u}{\vartheta}+\overset{u+u}{\vartheta}\ .\ \overset{u-u'}{\vartheta}\}$ (Suffixes 1.)

7	0	0	7	=	$M-iQ$	.	$M'+iQ'$	+	$N+iP$	. $N'-iP'$
3	4	4	3		$-i.M+iQ$		,,	+	$i.N-iP$	,,
15	8	8	15		$i.M+iQ$		,,	+	$i.N-iP$	,,
11	12	12	11		$M-iQ$		,,	−	$.N+iP$	,,
6	1	1	6		$N-iP$		,,	+	$M+iQ$	,,
2	5	5	2		$-i.N-iP$		,,	+	$i.M-iQ$	,,
14	9	9	14		$+i.N+iP$		,,	+	$i.M-iQ$	,,
10	13	13	10		$N-iP$		,,	−	$.M+iQ$	,,

$\frac{1}{2}\{\overset{u+u'}{\vartheta}\ .\ \overset{u-u'}{\vartheta}-\overset{u+u'}{\vartheta}\ .\ \overset{u-u'}{\vartheta}\}$ (Suffixes 1.)

7	0	0	7	=	$M+iQ$	.	$M'-iQ'$	+	$N-iP$	. $N'+iP'$
3	4	4	3		$-i.M-iQ$		,,	+	$i.N+iP$	,,
15	8	8	15		$-i.M-iQ$		,,	+	$i.N+iP$	,,
11	12	12	11		$M+iQ$		,,	−	$.N+iP$	,,
6	1	1	6		$N+iP$		,,	+	$M-iQ$	,,
2	5	5	2		$-i.N-iP$		,,	+	$i.M+iQ$	,,
14	9	9	14		$+i.N-iP$		,,	+	$i.M+iQ$	,,
10	13	13	10		$N+iP$		,,	−	$.M-iQ$	,,

104. FIFTEENTH set of 16.

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta} + \overset{u+u'}{\vartheta} \,.\, \overset{u-u}{\vartheta}\}$ (Suffixes 2.)

11	0	0	11	=	$M-iQ$	$.\ M'+iQ'$	$+$	$N-iP$	$.\ N'+iP'$
15	4	4	15		$i.M+iQ$	,,	$-$	$i.N+iP$	,,
3	8	8	3		$-i.M+iQ$	,,	$-$	$i.N+iP$	,,
7	12	12	7		$M-iQ$	,,	$-$	$.N-iP$	,,
10	1	1	10		$N-iP$	,,	$+$	$M-iQ$	,,
14	5	5	14		$-i.N+iP$	,,	$+$	$i.M+iQ$	,,
2	9	9	2		$-i.N+iP$	,,	$-$	$i.M+iQ$	,,
6	13	13	6		$-\ .N-iP$	,,	$+$	$M-iQ$	,,

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta} - \overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta}\}$ (Suffixes 2.)

11	0	0	11	=	$M+iQ$	$.\ M'-iQ'$	$+$	$N+iP$	$.\ N'-iP'$
15	4	4	15		$i.M-iQ$	,,	$-$	$i.N-iP$	,,
3	8	8	3		$-i.M-iQ$	,,	$-$	$i.N-iP$	,,
7	12	12	7		$M+iQ$	,,	$-$	$.N+iP$	,,
10	1	1	10		$N+iP$	,,	$+$	$M+iQ$	,,
14	5	5	14		$-i.N-iP$	,,	$+$	$i.M-iQ$	,,
2	9	9	2		$-i.N-iP$	,,	$-$	$i.M-iQ$	,,
6	13	13	6		$-\ .N+iP$	,,	$+$	$M+iQ$	,,

105. SIXTEENTH set of 16.

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta} + \overset{u+u'}{\vartheta} \,.\, \overset{u-u}{\vartheta}\}$ (Suffixes 3.)

15	0	0	15	=	$M-Q$	$.\ M'-Q'$	$+$	$N+P$	$.\ N'+P'$
11	4	4	11		$-i.M+Q$	,,	$+$	$i.N-P$	,,
7	8	8	7		$-i.M+Q$	,,	$-$	$i.N-P$	,,
3	12	12	3		$-\ .M-Q$	,,	$+$	$N+P$	,,
14	1	1	14		$N-P$	,,	$+$	$M+Q$	,,
10	5	5	10		$-i.N+P$	,,	$+$	$i.M-Q$	,,
6	9	9	6		$-i.N+P$	,,	$-$	$i.M-Q$	,,
2	13	13	2		$-\ .N-P$	,,	$+$	$M+Q$	,,

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta} - \overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta}\}$ (Suffixes 3.)

15	0	0	15	=	$M+Q$	$.\ M'+Q'$	$+$	$N-P$	$.\ N'-P'$
11	4	4	11		$-i.M-Q$	,,	$+$	$i.N+P$	,,
7	8	8	7		$-i.M-Q$	,,	$-$	$i.N+P$	,,
3	12	12	3		$-\ .M+Q$	,,	$+$	$N-P$	,,
14	1	1	14		$N+P$	,,	$+$	$M-Q$	,,
10	5	5	10		$-i.N-P$	,,	$+$	$i.M+Q$	,,
6	9	9	6		$-i.N-P$	,,	$-$	$i.M+Q$	,,
2	13	13	2		$-\ .N+P$	,,	$+$	$M-Q$	,,

106. In the square set, writing $u'=v'=0$, and $\alpha, \beta, \gamma, \delta$ for X′, Y′, Z′, W′; also slightly altering the arrangement,

the system becomes

u			X	Y	Z	W
ϑ^2	0	=	α	β	γ	δ
	4	=	α	$-\beta$	γ	$-\delta$
	8	=	α	β	$-\gamma$	$-\delta$
	12	=	α	$-\beta$	$-\gamma$	δ
	1	=	β	α	δ	γ
	5	=	β	$-\alpha$	δ	$-\gamma$
	9	=	β	α	$-\delta$	$-\gamma$
	13	=	β	$-\alpha$	$-\delta$	γ
	2	=	γ	δ	α	β
	6	=	γ	$-\delta$	α	$-\beta$
	10	=	γ	δ	$-\alpha$	$-\beta$
	14	=	γ	$-\delta$	$-\alpha$	β
	3	=	δ	γ	β	α
	7	=	δ	$-\gamma$	β	$-\alpha$
	11	=	δ	γ	$-\beta$	$-\alpha$
	15	=	δ	$-\gamma$	$-\beta$	α

and further writing herein $u=0$, $v=0$ it becomes

	0				c^2
ϑ^2	0	=	$\alpha^2+\beta^2+\gamma^2+\delta^2$	=	0
	4	=	$\alpha^2-\beta^2+\gamma^2-\delta^2$	=	4
	8	=	$\alpha^2-\beta^2-\gamma^2-\delta^2$	=	8
	12	=	$\alpha^2-\beta^2-\gamma^2+\delta^2$	=	12
	1	=	$2(\alpha\beta+\gamma\delta)$	=	1
	5		0		
	9	=	$2(\alpha\beta-\gamma\delta)$	=	9
	13		0		
	2	=	$2(\alpha\gamma+\beta\delta)$	=	2
	6	=	$2(\alpha\gamma-\beta\delta)$	=	6
	10		0		
	14		0		
	3	=	$2(\alpha\delta+\beta\gamma)$	=	3
	7		0		
	11		0		
	15	=	$2(\alpha\delta-\beta\gamma)$	=	15

viz.: this last is the before-mentioned system of equations giving the values of the 10 zero-functions c in terms of the four constants $\alpha, \beta, \gamma, \delta$.

107. The system first obtained is a system of 16 equations

$$\vartheta_0{}^2(u, v)=\alpha X+\beta Y+\gamma Z+\delta W, \text{ \&c.}$$

showing that the squares of the theta-functions are each of them a linear function of the four quantities X, Y, Z, W. If the functions on the right hand side were independent (asyzygetic) linear functions of (X, Y, Z, W) it would follow that any four (selected at pleasure) of the squared theta-functions were linearly independent, and that we could in terms of these four express linearly each of the remaining 12 squared functions. But this is not so; the form of the linear functions of (X, Y, Z, W) is such that we can (and that in 16 different ways) select out of the 16 linear functions six functions, such that any four of them are connected by a linear equation; and there are consequently 16 hexads of squared theta-functions, such that any four out of the same hexad are connected by a linear relation. The hexads are shown by the foregoing "Table of the 16 KUMMER hexads."

108. The *à posteriori* verification is immediately effected; taking for instance the first column, the equations are

	$\vartheta^2 u$		X	Y	Z	W
A	11	=	δ	γ	$-\beta$	$-\alpha,$
B	7		δ	$-\gamma$	β	$-\alpha,$
AB	6		γ	$-\delta$	α	$-\beta,$
CD	2		γ	δ	α	$\delta,$
CE	1		β	α	δ	$\gamma,$
DE	9		β	α	δ	$-\gamma.$

viz.: it should thence follow that there is a linear relation between any four of the six squared functions 11, 7, 6, 2, 1, 9: and it is accordingly seen that this is so. It further appears that in the several linear relations, the coefficients (obtained in the first instance as functions of α, β, γ, δ) are in fact the 10 constants c: the 15 relations connecting the several systems of four out of the six squared functions are given in the table.

Read

$$\cdot \quad c_6^2\vartheta_6^2 - c_2^2\vartheta_2^2 + c_1^2\vartheta_1^2 - c_9^2\vartheta_9^2 = 0,$$

$$c_6^2\vartheta_{11}^2 \quad \cdot \quad + c_{15}^2\vartheta_2^2 - c_{12}^2\vartheta_1^2 + c_4^2\vartheta_9^2 = 0, \quad \&c.$$

109.

ϑ^2	11	7	6	2	1	9	= 0.
c^2			6	− 2	1	− 9	
	6			+15	−12	+ 4	
	− 2		−15		+ 8	− 0	
	1		+12	− 8		+ 3	
	− 9		− 4	+ 0	− 3		
		6		3	− 0	+ 8	
		− 2	− 3		+ 4	−12	
		1	+ 0	− 4		−15	
		− 9	− 8	+12	+15		
	−15	+ 3	+ 2	− 6			
	−12	+ 0	+ 1		− 6		
	− 4	+ 8	+ 9			− 6	
	− 3	+15			+ 9	− 1	
	− 0	+12		+ 9		− 2	
	− 8	+ 4		+ 1	− 2		

110. The first set of 16 equations is the square-set, which has been already considered. If in each of the other sets of 16 equations we write in like manner $u'=0$, each set in fact reduces itself to eight equations; sets 2, 3, 4 give thus $8+8+8$, $=24$ equations; sets 5 to 8, 9 to 12, and 13 to 15, give each $8+8+8+8$, $=32$ equations; or we have sets of 24, 32, 32, 32, together 120 equations, the number being of course one half of $256-16$, the number of equations after deducting the 16 equations of the square set.

111. THE first set, 24 equations.

This is derived from the second, third, and fourth sets, each of 16 equations, by writing therein $u'=0$. Taking α_1, β_1, γ_1, δ_1 for the zero-functions corresponding to X_1, Y_1, Z_1, W_1, then on writing $u'=0$, X_1', Y_1', Z_1', W_1' become α_1, β_1, γ_1, δ_1. In the second set of 16 equations, the first equations thus are

$$\begin{array}{ll} \vartheta_4 u\,.\,\vartheta_0 u=\alpha_1X_1+\gamma_1Z_1, & \qquad 0=\beta_1Y_1+\delta_1W_1, \\ \vartheta_{12}u\ \ \vartheta_8 u=\alpha_1X_1-\gamma_1Z_1, & \qquad 0=\beta_1Y_1-\delta_1W_1, \\ \qquad\quad \vdots & \qquad\quad \vdots \end{array}$$

viz., the equations of the column require that, and are all satisfied if, $\beta_1=0$, $\delta_1=0$: hence the zero functions are α_1, 0, γ_1, 0; and this being so we have only the equations of the first column. And similarly as regards the third and fourth sets; the zero values corresponding to

$$\begin{array}{l|l|l} & X_1,\ Y_1,\ Z_1,\ W_1 & X_2,\ Y_2,\ Z_2,\ W_2 & X_3,\ Y_3,\ Z_3,\ W_3 \\ \text{are} & \alpha_1\quad 0\quad \gamma_1\quad 0 & \alpha_2\quad \beta_2\quad 0\quad 0 & \alpha_3\quad 0\quad 0\quad \delta_3\,; \end{array}$$

and we have in all $8+8+8$, $=24$ equations. These are

ϑu	. ϑu		(Suffixes 1.) X	Z
4	0	=	α	γ
12	8	=	α	$-\gamma$
6	2	=	γ	α
14	10	=	γ	$-\alpha$
			Y	W
5	1	=	α	γ
13	9	=	α	$-\gamma$
7	3	=	γ	α
15	11	=	γ	$-\alpha$
$\vartheta 0$	. $\vartheta 0$			
4	0	=	$\alpha^2+\gamma^2$	
12	8	=	$\alpha^2-\gamma^2$	
6	2	=	$2\alpha\gamma$	

ϑu	. ϑu		(Suffixes 2.) X	Y
8	0	=	α	β
12	4	=	α	$-\beta$
9	1	=	β	α
13	5	=	β	$-\alpha$
			Z	W
10	2	=	α	β
14	6	=	α	$-\beta$
11	3	=	β	α
15	7	=	β	$-\alpha$
$\vartheta 0$	. $\vartheta 0$			
8	0	=	$\alpha^2+\beta^2$	
12	4	=	$\alpha^2-\beta^2$	
9	1	=	$2\alpha\beta$	

ϑu	. ϑu		(Suffixes 3.) X	W
12	0	=	α	δ
8	4	=	α	$-\delta$
15	3	=	δ	α
11	7	=	$-\delta$	α
			Y	Z
13	1	=	α	δ
9	5	=	α	$-\delta$
14	2	=	δ	α
10	6	=	$-\delta$	α
$\vartheta 0$	. $\vartheta 0$			
12	0	=	$\alpha^2+\delta^2$	
8	4	=	$\alpha^2-\delta^2$	
15	3	=	$2\alpha\delta$	

112. THE second set, 32 equations.

To exhibit these in a convenient form I alter the notation, viz., I write

$$\begin{array}{lcccc|cccc|} & \mathrm{E}+\mathrm{G}, & i(\mathrm{E}-\mathrm{G}), & (\mathrm{F}+\mathrm{H}), & i(\mathrm{F}-\mathrm{H}) & \mathrm{E}_1+i\mathrm{G}_1, & \mathrm{E}_1-i\mathrm{G}_1, & \mathrm{F}_1+i\mathrm{H}_1, & \mathrm{F}_1-i\mathrm{H}_1 \\ = & \mathrm{X}, & \mathrm{Y}, & \mathrm{Z}, & \mathrm{W} & \mathrm{X}_1, & \mathrm{Y}_1, & \mathrm{Z}_1, & \mathrm{W}_1 \end{array}$$

$$\begin{array}{lcccc|cccc} & (\mathrm{E}_2+\mathrm{G}_2), & i(\mathrm{E}_2-\mathrm{G}_2), & (\mathrm{F}_2+\mathrm{H}_2), & i(\mathrm{F}_2-\mathrm{H}_2) & \mathrm{E}_3+i\mathrm{G}_3, & \mathrm{E}_3-i\mathrm{G}_3, & \mathrm{F}_3+i\mathrm{H}_3, & \mathrm{F}_3-i\mathrm{H}_3 \\ = & \mathrm{X}_2, & \mathrm{Y}_2, & \mathrm{Z}_2, & \mathrm{W}_2 & \mathrm{X}_3, & \mathrm{Y}_3, & \mathrm{Z}_3, & \mathrm{W}_3, \end{array}$$

so that as regards the present set of equations, X, Y, X, &c., signify as just mentioned. And this being so the corresponding zero-values are

$$\alpha,\quad 0,\quad \gamma,\quad 0 \quad\Big|\quad \alpha_1,\quad 0,\quad \gamma_1,\quad 0 \quad\Big|\quad \alpha_2,\quad 0,\quad 0,\quad \delta_2 \quad\Big|\quad \alpha_3,\quad 0,\quad 0,\quad \delta_3.$$

The equations then are

(Suffixes 0.)	(Suffixes 1.)	(Suffixes 2.)	(Suffixes 3.)
$\vartheta u\,.\,\vartheta u \quad \overbrace{\mathrm{X}\quad\mathrm{Z}}$	$\vartheta u\,.\,\vartheta u \quad \overbrace{\mathrm{X}\quad\mathrm{Z}}$	$\vartheta u\,.\,\vartheta u \quad \overbrace{\mathrm{X}\quad\mathrm{W}}$	$\vartheta u\,.\,\vartheta u \quad \overbrace{\mathrm{X}\quad\mathrm{W}}$
$1 \quad 0 = \alpha \quad \gamma$	$1 \quad 4 = -i\alpha \quad -i\gamma$	$9 \quad 0 = \alpha \quad -\delta$	$9 \quad 4 = -i\alpha \quad -i\delta$
$9 \quad 8 = \alpha \quad -\gamma$	$9 \quad 12 = -i\alpha \quad +i\gamma$	$1 \quad 8 = \alpha \quad \delta$	$1 \quad 12 = -i\alpha \quad +i\delta$
$3 \quad 2 = \gamma \quad \alpha$	$3 \quad 6 = -i\gamma \quad -i\alpha$	$15 \quad 6 = \delta \quad \alpha$	$15 \quad 2 = \delta \quad \alpha$
$11 \quad 10 = \gamma \quad -\alpha$	$11 \quad 14 = -i\gamma \quad +i\alpha$	$7 \quad 14 = -\delta \quad \alpha$	$7 \quad 10 = -\delta \quad \alpha$
$\overbrace{\mathrm{Y}\quad\mathrm{W}}$	$\overbrace{\mathrm{Y}\quad\mathrm{W}}$	$\overbrace{\mathrm{Y}\quad\mathrm{Z}}$	$\overbrace{\mathrm{Y}\quad\mathrm{Z}}$
$5 \quad 4 = \alpha \quad \gamma$	$5 \quad 0 = \alpha \quad \gamma$	$13 \quad 14 = \alpha \quad \delta$	$13 \quad 0 = \alpha \quad \delta$
$13 \quad 12 = \alpha \quad -\gamma$	$13 \quad 8 = \alpha \quad -\gamma$	$5 \quad 12 = \alpha \quad -\delta$	$5 \quad 8 = \alpha \quad -\delta$
$7 \quad 6 = \gamma \quad \alpha$	$7 \quad 2 = \gamma \quad \alpha$	$11 \quad 2 = -\delta \quad \alpha$	$11 \quad 6 = -i\delta \quad -i\alpha$
$15 \quad 14 = \gamma \quad -\alpha$	$15 \quad 10 = \gamma \quad -\alpha$	$3 \quad 10 = \delta \quad \alpha$	$3 \quad 14 = i\delta \quad -i\alpha$
$\vartheta 0\,.\,\vartheta 0$	$\vartheta 0\,.\,\vartheta 0$	$\vartheta 0\,.\,\vartheta 0$	$\vartheta 0\,.\,\vartheta 0$
$1 \quad 0 = \alpha^2+\gamma^2$	$1 \quad 4 = -i(\alpha^2+\gamma^2)$	$9 \quad 0 = \alpha^2-\delta^2$	$9 \quad 4 = -i(\alpha^2+\delta^2)$
$9 \quad 8 = \alpha^2-\gamma^2$	$9 \quad 12 = -i(\alpha^2-\gamma^2)$	$1 \quad 8 = \alpha^2+\delta^2$	$1 \quad 12 = -i(\alpha^2-\delta^2)$
$3 \quad 2 = 2\alpha\gamma$	$3 \quad 6 = -\ 2i\alpha\gamma$	$15 \quad 6 = 2\alpha\delta$	$15 \quad 2 = 2\alpha\delta$

113. THIRD set, 32 equations.

We again change the notation, writing

$$\begin{array}{rcccc|cccc|}
 & \mathrm{I}+\mathrm{K}, & i(\mathrm{I}-\mathrm{K}), & \mathrm{J}+\mathrm{L}, & i(\mathrm{J}-\mathrm{L}) & \mathrm{I}_1+\mathrm{K}_1, & i(\mathrm{I}_1-\mathrm{K}_1), & (\mathrm{J}_1+\mathrm{L}_1), & i(\mathrm{J}_1-\mathrm{L}_1) \\
= & \mathrm{X}, & \mathrm{Y}, & \mathrm{Z}, & \mathrm{W} & \mathrm{X}_1, & \mathrm{Y}_1, & \mathrm{Z}_1, & \mathrm{W}_1
\end{array}$$

$$\begin{array}{rcccc|cccc}
 & \mathrm{I}_2+i\mathrm{K}_2, & \mathrm{I}_2-i\mathrm{K}_2, & \mathrm{J}_2+i\mathrm{L}_2, & \mathrm{J}_2-i\mathrm{L}_2 & \mathrm{I}_3+i\mathrm{K}_3, & \mathrm{I}_3-i\mathrm{K}_3, & \mathrm{J}_3+i\mathrm{L}_3, & \mathrm{J}_3-i\mathrm{L}_3 \\
= & \mathrm{X}_2, & \mathrm{Y}_2, & \mathrm{Z}_2, & \mathrm{W}_2 & \mathrm{X}_3. & \mathrm{Y}_3, & \mathrm{Z}_3, & \mathrm{W}_3
\end{array}$$

the zero values being

$$\alpha,\ 0,\ \gamma,\ 0 \;\big|\; \alpha_1,\ 0,\ 0,\ \delta_1 \;\big|\; \alpha_2,\ 0,\ \gamma_2,\ 0 \;\big|\; \alpha_3,\ 0,\ 0,\ \delta_3$$

Then equations are

(Suffixes 0.)	(Suffixes 1.)	(Suffixes 2.)	(Suffixes 3.)
$\vartheta u\,.\,\vartheta u \quad \overbrace{\mathrm{X}\quad\mathrm{Z}}$	$\vartheta u\,.\,\vartheta u \quad \overbrace{\mathrm{X}\quad\mathrm{W}}$	$\vartheta u\,.\,\vartheta u \quad \overbrace{\mathrm{X}\quad\mathrm{Z}}$	$\vartheta u\,.\,\vartheta u \quad \overbrace{\mathrm{X}\quad\mathrm{W}}$
$2 \quad 0 = \alpha \quad \gamma$	$6 \quad 0 = \alpha \quad -\delta$	$2 \quad 8 = -i\alpha \quad -i\gamma$	$6 \quad 8 = -i\alpha \quad -i\delta$
$6 \quad 4 = \alpha \quad -\gamma$	$2 \quad 4 = \alpha \quad \delta$	$6 \quad 12 = -i\alpha \quad +i\gamma$	$2 \quad 12 = -i\alpha \quad i\delta$
$3 \quad 1 = \gamma \quad \alpha$	$15 \quad 9 = \delta \quad \alpha$	$3 \quad 9 = -i\gamma \quad -i\alpha$	$15 \quad 1 = \delta \quad \alpha$
$7 \quad 5 = \gamma \quad -\alpha$	$11 \quad 13 = -\delta \quad \alpha$	$7 \quad 13 = -i\gamma \quad +i\alpha$	$11 \quad 5 = -\ \delta \quad \alpha$
$\overbrace{\mathrm{Y}\quad\mathrm{W}}$	$\overbrace{\mathrm{Y}\quad\mathrm{Z}}$	$\overbrace{\mathrm{Y}\quad\mathrm{W}}$	$\overbrace{\mathrm{Y}\quad\mathrm{Z}}$
$10 \quad 8 = \alpha \quad \gamma$	$14 \quad 8 = \alpha \quad \delta$	$10 \quad 0 = \alpha \quad \gamma$	$14 \quad 0 = \alpha \quad \delta$
$14 \quad 12 = \alpha \quad -\gamma$	$10 \quad 12 = \alpha \quad -\delta$	$14 \quad 4 = \alpha \quad -\gamma$	$10 \quad 4 = \alpha \quad -\ \delta$
$11 \quad 9 = \gamma \quad \alpha$	$7 \quad 1 = -\delta \quad \alpha$	$11 \quad 1 = \gamma \quad \alpha$	$7 \quad 9 = -i\delta \quad -i\alpha$
$15 \quad 13 = \gamma \quad -\alpha$	$3 \quad 5 = \delta \quad \alpha$	$15 \quad 5 = \gamma \quad -\alpha$	$3 \quad 13 = i\delta \quad -i\alpha$
$\vartheta 0\,.\,\vartheta 0$	$\vartheta 0\,.\,\vartheta 0$	$\vartheta 0\,.\,\vartheta 0$	$\vartheta 0\,.\,\vartheta 0$
$2 \quad 0 = \alpha^2+\gamma^2$	$6 \quad 0 = \alpha^2-\delta^2$	$2 \quad 8 = -i(\alpha^2+\gamma^2)$	$6 \quad 8 = -i(\alpha^2+\delta^2)$
$6 \quad 4 = \alpha^2-\gamma^2$	$2 \quad 4 = \alpha^2+\delta^2$	$6 \quad 12 = -i(\alpha^2-\gamma^2)$	$2 \quad 12 = -i(\alpha^2-\delta^2)$
$3 \quad 1 = 2\alpha\gamma$	$15 \quad 9 = 2\alpha\delta$	$3 \quad 9 = -2i\alpha\gamma$	$15 \quad 1 = 2\alpha\delta$

114. FOURTH set, 32 equations.

Again changing the notation we write

$$\begin{array}{rcccc|cccc} & M_2+iQ_2, & M_2-iQ_2, & N_2+iP_2, & N_2-iP_2 & M_3+Q_3, & i(M_3-Q_3), & N_3+P_3, & i(N_3-P_3) \\ = & X, & Y, & Z, & W & X_1, & Y_1, & Z_1, & W_1, \\ & & & & & & & & \\ & M+Q, & i(M-Q), & N+P, & i(N-P) & M_1+iQ_1, & M_1-iQ_1, & N_1+iP_1, & N_1-iP_1 \\ = & X_2, & Y_2, & Z_2, & W_2 & X_3, & Y_3, & Z_3, & W_3 \end{array}$$

the zero values being

$$\alpha,\ 0,\ \gamma,\ 0 \;\big|\; \alpha_1,\ 0,\ 0_1,\ \delta \;\big|\; \alpha_2,\ 0,\ \gamma_2,\ 0 \;\big|\; 0,\ \beta_3,\ \gamma_3,\ 0$$

and the equations then are

(Suffixes 0.)

ϑu .	ϑu		X	Z
0	3	=	α	γ
15	12	=	$-\alpha$	γ
2	1	=	γ	α
14	13	=	$-\gamma$	α
			Y	W
4	7	=	α	$-\gamma$
8	11	=	α	γ
6	5	=	γ	$-\alpha$
10	9	=	γ	α

$\vartheta 0$.	$\vartheta 0$		
0	3	=	$\alpha^2+\gamma^2$
15	12	=	$-(\alpha^2-\gamma^2)$
2	1	=	$2\alpha\gamma$

(Suffixes 1.)

ϑu .	ϑu		X	W
3	4	=	$-i\alpha$	$i\delta$
15	8	=	$i\alpha$	$i\delta$
6	1	=	δ	α
10	13	=	$-\delta$	α
			Y	Z
7	0	=	α	δ
11	12	=	α	$-\delta$
2	5	=	$i\delta$	$-i\alpha$
14	9	=	$i\delta$	$i\alpha$

$\vartheta 0$.	$\vartheta 0$		
3	4	=	$-i(\alpha^2-\delta^2)$
15	8	=	$i(\alpha^2+\delta^2)$
6	1	=	$2\alpha\delta$

(Suffixes 2.)

ϑu .	ϑu		X	Z
15	4	=	$i\alpha$	$-i\gamma$
3	8	=	$-i\alpha$	$-i\gamma$
14	5	=	$i\gamma$	$-i\alpha$
2	9	=	$-i\gamma$	$-i\alpha$
			Y	W
11	0	=	α	γ
7	12	=	α	$-\gamma$
10	1	=	γ	α
6	13	=	γ	$-\alpha$

$\vartheta 0$.	$\vartheta 0$		
15	4	=	$i(\alpha^2-\gamma^2)$
3	8	=	$-i(\alpha^2+\gamma^2)$
2	9	=	$-\ 2i\alpha\gamma$

(Suffixes 3.)

ϑu .	ϑu		Y	Z
15	0	=	$-\beta$	γ
3	12	=	β	γ
10	5	=	γ	$-\beta$
6	9	=	$-\gamma$	$-\beta$
			X	W
11	4	=	$-\beta$	γ
7	8	=	$-\beta$	$-\gamma$
14	1	=	γ	$-\beta$
2	13	=	γ	β

$\vartheta 0$.	$\vartheta 0$		
15	0	=	$-(\beta^2-\gamma^2)$
3	12	=	$\beta^2+\gamma^2$
6	9	=	$-\ 2\beta\gamma$

115. It will be noticed that the pairs of theta-functions which present themselves in these equations are the same as in the foregoing "Table of the 120 pairs." And the equations show that the four products, each of a pair of theta-functions, belonging to the upper half or to the lower half of any column of the table, are such that any three of the four products are connected by a linear equation. The coefficients of these linear relations are, in fact, functions such as the $\alpha^2+\delta^2$, $\alpha^2-\delta^2$, $2\alpha\delta$ written down at the foot of the several systems of eight equations, and they are consequently products each of two zero-functions c.

Thus (see "The first set, 24 equations") we have

$$
\begin{array}{cccc|cccc|ccc}
 & & \multicolumn{2}{c|}{\text{(Suffixes 3.)}} & & & \multicolumn{2}{c|}{\text{(Suffixes 3.)}} & & & \text{(Suffixes 3.)} \\
\vartheta u & .\ \vartheta u & \multicolumn{2}{c|}{\overbrace{\mathrm{X} \quad \mathrm{W}}} & \vartheta u & .\ \vartheta u & \multicolumn{2}{c|}{\overbrace{\mathrm{Y} \quad \mathrm{Z}}} & \vartheta 0 & .\ \vartheta 0 & \\
4 & 8 = & \alpha & -\delta & 5 & 9 = & \alpha & -\delta & 4 & 8 & =\alpha^2-\delta^2 \\
0 & 12 = & \alpha & \delta & 1 & 13 = & \alpha & \delta & 0 & 12 & =\alpha+\delta^2 \\
3 & 15 = & \delta & \alpha & 2 & 14 = & \delta & \alpha & 15 & 3 & =2\alpha\delta \\
7 & 11 = & -\delta & \alpha & 6 & 10 = & -\delta & \alpha & & &
\end{array}
$$

116. In the left hand four of these, omitting successively the first, second, third, and fourth equation, and from the remaining three eliminating the X_3 and W_3, we write down, almost mechanically,

$$
\begin{array}{cc|cccc|}
\vartheta u & .\ \vartheta u & & & & \\
4 & 8 & & +2\alpha\delta, & -\delta^2-\alpha^2, & \alpha^2-\delta^2 \\
0 & 12 & -2\alpha\delta, & . & -\delta^2+\alpha^2, & \alpha^2+\delta^2 \\
3 & 15 & \alpha^2+\delta^2, & \delta^2-\alpha^2, & . & 2\alpha\delta \\
7 & 11 & -\alpha^2+\delta^2, & \delta^2+\alpha^2, & -2\alpha\delta &
\end{array}
$$

and thence derive the first of the next following system of equations; read

$$
\begin{array}{rrrr}
 & c_3c_{15}\vartheta_0\vartheta_{12} & -c_0c_{12}\vartheta_3\vartheta_{15} & +c_4c_8\,\vartheta_7\vartheta_{11}=0, \\
-c_3c_{15}\vartheta_4\vartheta_8 & & +c_4c_8\,\vartheta_3\vartheta_{15} & -c_0c_{12}\vartheta_7\vartheta_{11}=0, \\
c_0c_{12}\vartheta_4\vartheta_8 & -c_4c_8\,\vartheta_0\vartheta_{12} & & +c_3c_{15}\vartheta_7\vartheta_{11}=0, \\
-c_4c_8\,\vartheta_4\vartheta_8 & +c_0c_{12}\vartheta_0\vartheta_{12} & -c_3c_{15}\vartheta_3\vartheta_{15} & =0,
\end{array}
$$

where the theta-functions have the arguments u, v.

Observe that on writing herein $u=0$, $v=0$, the first three equations become each of them identically $0=0$; the fourth equation becomes $-c_4{}^2c_8{}^2+c_0{}^2c_{12}{}^2-c_3{}^2c_{15}{}^2=0$, which is one of the relations between the c's, and which serves as a verification.

But in the right hand system, on writing $u=v=0$, each of the four equations becomes identically $0=0$.

117. The equations are

ϑ	4.8	0.12	3.15	7.11	=0,
c		3.15	−0.12	4.8	
	−3.15		−4.8	−0.12	
	0.12	−4.8		3.15	
	−4.8	0.12	−3.15		

ϑ	5.9	1.13	2.14	6.10	=0,
c		3.15	−0.12	4.8	
	−3.15		4.8	−0.12	
	0.12	−4.8		3.15	
	−4.8	−0.12	−3.15		

ϑ	6.8	2.12	1.15	5.11	=0,
c		1.15	−2.12	6.8	
	−1.15		6.8	−2.11	
	2.12	−6.8		1.15	
	−6.8	2.11	−1.15		

ϑ	7.9	3.13	0.14	4.10	=0,
c		1.15	−2.12	6.8	
	−1.15		6.8	−2.12	
	2.12	−6.8		1.15	
	−6.8	2.12	−1.15		

ϑ	0.6	2.4	9.15	11.13	=0,
c		9.15	−2.4	0.6	
	−9.15		0.6	−2.4	
	2.4	−0.6		9.15	
	−0.6	2.4	−9.15		

ϑ	1.7	3.5	8.14	10.12	=0,
c		9.15	−2.4	0.6	
	−9.15		0.6	−2.4	
	2.4	−0.6		9.15	
	−0.6	2.4	−9.15		

ϑ	3.6	1.4	9.12	14.11	=0,
c		9.12	−1.4	3.6	
	−9.12		3.6	−1.4	
	1.4	−3.6		9.12	
	−3.6	1.4	−9.12		

ϑ	2.7	3.5	0.5	8.13	=0,
c		9.12	−1.4	3.6	
	−9.12		3.6	−1.4	
	1.4	−3.6		9.12	
	−3.6	1.4	−9.12		

ϑ	8.9	0.1	2.3	10.11	=0,
c		−2.3	0.1	8.9	
	2.3		−8.9	−0.1	
	−0.1	8.9		2.3	
	−8.9	0.1	−2.3		

ϑ	12.13	4.5	6.7	14.15	=0,
c		−2.3	0.1	8.9	
	2.3		−8.9	−0.1	
	−0.1	8.9		2.3	
	−8.9	0.1	−2.3		

ϑ	4.6	0.2	1.3	5.7	=0,
c		−1.3	2.0	4.6	
	1.3		−4.6	−0.2	
	−0.2	4.6		1.3	
	−4.6	0.2	−1.3		

ϑ	9.11	13.15	12.14	8.10	=0,
c		−1.3	0.2	4.6	
	1.3		−4.6	−0.2	
	−0.2	−4.6		1.3	
	4.6	0.2	−1.3		

ϑ	6.12	2.8	3.9	7.13	=0,
c		3.9	−2.8	−6.12	
	−3.9		6.12	2.8	
	2.8	−6.12		−3.9	
	6.12	−2.8	3.9		

ϑ	1.11	5.15	4.14	0.10	=0,
c		3.9	−2.8	6.12	
	−3.9		−6.12	2.8	
	2.8	6.12		−3.9	
	−6.12	−2.8	3.9		

ϑ	6.15	1.8	0.9	7.14	=0,
c		0.9	−1.8	−6.15	
	−0.9		6.15	1.8	
	1.8	−6.15		−0.9	
	6.15	−1.8	0.9		

ϑ	2.11	5.12	4.13	3.10	=0,
c		0.9	−1.8	6.15	
	−0.9		−6.15	1.8	
	1.8	6.15		−0.9	
	−6.15	−1.8	0.9		

ϑ	4.9	1.12	2.15	7.10	=0,
c		2.15	−1.12	4.9	
	−2.15		4.9	−1.12	
	1.12	−4.9		2.15	
	−4.9	1.12	−2.15		

ϑ	0.13	5.8	6.11	3.14	=0,
c		2.15	1.12	−4.9	
	−2.15		−4.9	1.12	
	−1.12	4.9		2.15	
	4.9	−1.12	−2.15		

ϑ	4.12	0.8	1.9	5.13	=0,
c		−1.9	0.8	4.12	
	1.9		−4.12	−0.8	
	−0.8	4.12		1.9	
	−4.12	0.8	−1.9		

ϑ	3.11	7.15	6.14	2.10	=0,
c		1.9	−0.8	4.12	
	−1.9		−4.12	0.8	
	0.8	4.12		−1.9	
	−4.12	−0.8	1.9		

ϑ	4.15	3.8	2.9	5.14	=0,
c		−2.9	3.8	4.15	
	2.9		−4.15	−3.8	
	−3.8	4.15		2.9	
	−4.15	3.8	−2.9		

ϑ	0.11	7.12	6.13	1.10	=0,
c		−2.9	3.8	−4.15	
	2.9		4.15	−3.8	
	−3.8	−4.15		2.9	
	4.15	3.8	−2.9		

ϑ	6.9	3.12	0.15	5.10	=0,
c		−0.15	3.12	−6.9	
	0.15		−6.9	3.12	
	−3.12	6.9		−0.15	
	6.9	−3.12	0.15		

ϑ	2.13	7.8	4.11	1.14	=0,
c		0.15	3.12	−6.9	
	−0.15		−6.9	3.12	
	−3.12	6.9		0.15	
	6.9	−3.12	−0.15		

ϑ	12.15	0.3	1.2	13.14	=0,
c		1.2	−0.3	−12.15	
	−1.2		12.15	0.3	
	0.3	−12.15		−1.2	
	12.15	−0.3	1.2		

ϑ	8.11	4.7	5.6	9.10	=0,
c		1.2	−0.3	12.15	
	−1.2		−12.15	0.3	
	0.3	12.15		−1.2	
	−12.15	−0.3	1.2		

ϑ	1.6	3.4	8.15	10.13	=0,
c		8.15	−3.4	1.6	
	−8.15		1.6	−3.4	
	3.4	−1.6		8.15	
	−1.6	3.4	−8.15		

ϑ	2.5	0.7	11.12	9.14	=0,
c		−8.15	−3.4	1.6	
	8.15		1.6	−3.4	
	3.4	−1.6		−8.15	
	8.15	3.4	8.15		

ϑ	2.6	0.4	8.12	10.14	$=0,$
c		−8.12	0.4	−2.6	
	8.12		−2.6	0.4	
	−0.4	2.6		−8.12	
	2.6	−0.4	8.12		

ϑ	1.5	3.7	11.15	9.13	$=0.$
c		−8.12	−0.4	2.6	
	8.12		2.6	−0.4	
	0.4	−2.6		−8.12	
	−2.6	0.4	8.12		

118. The foregoing equations may be verified, and it is interesting to verify them, by means of the approximate values of the functions: thus for one of the equations we have

$$\begin{array}{l|l}
c_3c_{15}\vartheta_0\vartheta_{12} \;\; i.e. & (2\Lambda+2\Lambda')(-2\Lambda+2\Lambda') \qquad . \qquad 1 \qquad . \qquad 1 \\
-c_0c_{12}\vartheta_3\vartheta_{15} & - \quad 1 \quad . \quad 1 \quad . \quad 2\Lambda\cos\tfrac{1}{2}\pi(u+v)+2\Lambda'\cos\tfrac{1}{2}\pi(u-v). \\
 & \qquad\qquad\qquad -2\Lambda\cos\tfrac{1}{2}\pi(u+v)+2\Lambda'\cos\tfrac{1}{2}\pi(u-v) \\
+c_4c_8\vartheta_7\vartheta_{11} & + \quad 1 \quad . \quad 1 \quad . \; -2\Lambda\sin\tfrac{1}{2}\pi(u+v)-2\Lambda'\sin\tfrac{1}{2}\pi(u-v). \\
 & \qquad\qquad\qquad -2\Lambda\sin\tfrac{1}{2}\pi(u+v)+2\Lambda'\sin\tfrac{1}{2}\pi(u-v) \\
=0, & =0,
\end{array}$$

viz., the equation to be verified is here

$$\begin{array}{l}
-4\Lambda^2 \qquad\qquad\qquad +4\Lambda'^2 \\
+4\Lambda^2\cos^2\tfrac{1}{2}\pi(u+v) -4\Lambda'^2\cos^2\tfrac{1}{2}\pi(u-v) \\
+4\Lambda'^2\sin^2\tfrac{1}{2}\pi(u+v)-4\Lambda'^2\sin^2\tfrac{1}{2}\pi(u-v) \\
=0, \text{ which is right.}
\end{array}$$

119. In the equation

$$\begin{array}{r|l}
c_9c_{12}\vartheta_1\vartheta_4 \;\; i.e. & 2Q.1.2Q\cos\tfrac{1}{2}\pi u.1 \\
-c_1c_4\vartheta_9\vartheta_{12} & -2Q.1.2Q\cos\tfrac{1}{2}\pi u.1 \\
+c_3c_6\vartheta_{14}\vartheta_{11} & \\
=0, & =0\,;
\end{array}$$

this is right, but there is no verification as to the term $c_3c_6\vartheta_{14}\vartheta_{11}$; taking the more approximate values, the term in question taken negatively, that is $-c_3c_6\vartheta_{14}\vartheta_{11}$ is $=$

$$-(2\Lambda+2\Lambda').\quad 2S.\quad -2S\sin\tfrac{1}{2}\pi v.\quad -2\Lambda\sin\tfrac{1}{2}\pi(u+v)+2\Lambda'\sin\tfrac{1}{2}\pi(u-v),$$

which is $=$

$$-8S^2(\Lambda+\Lambda')^2\cos\tfrac{1}{2}\pi u+8S^2(\Lambda+\Lambda')\Lambda\cos\tfrac{1}{2}\pi(u+2v)+8S^2(\Lambda+\Lambda')\Lambda'\cos\tfrac{1}{2}\pi(u-2v)$$

and this ought therefore to be the value of the first two terms, that is of

$$\begin{aligned}&(2Q+2Q^9-2A-2A')(1-2Q^4-2S^4)\{2Q\cos\tfrac{1}{2}\pi u+2Q^9\cos\tfrac{3}{2}\pi u\\ &\qquad +2A\cos\tfrac{1}{2}\pi(u+2v)+2A'\cos\tfrac{1}{2}\pi(u-2v)\}(1-2Q^4\cos\pi u+2S^4\cos\pi v)\\ &-(2Q+2Q^9+2A+2A')(1-2Q^4+2S^4)\{2Q\cos\tfrac{1}{2}\pi u+2Q^9\cos\tfrac{3}{2}\pi u\\ &\qquad -2A\cos\tfrac{1}{2}\pi(u+2v)-2A'\cos\tfrac{1}{2}\pi(u-2v)\}(1-2Q^4\cos\pi u-2S^4\cos\pi v),\end{aligned}$$

which to the proper degree of approximation is

$$\begin{aligned}&=(2Q-4Q^5-4QS^4+2Q^9-2A-2A')\{2Q\cos\tfrac{1}{2}\pi u-4Q^5\cos\tfrac{1}{2}\pi u\cos\pi u\\ &\qquad +4QS^4\cos\tfrac{1}{2}\pi u\cos\pi v+2Q^9\cos\tfrac{3}{2}\pi u+2A\cos\tfrac{1}{2}\pi(u+2v)+2A'\cos\tfrac{1}{2}\pi(u-2v)\}\\ &-(2Q-4Q^5+4QS^4+2Q^9+2A+2A')\{2Q\cos\tfrac{1}{2}\pi u-4Q^5\cos\tfrac{1}{2}\pi u\cos\pi u\\ &\qquad -4QS^4\cos\tfrac{1}{2}\pi u\cos\pi v+2Q^9\cos\tfrac{3}{2}\pi u-2A\cos\tfrac{1}{2}\pi(u+2v)-2A'\cos\tfrac{1}{2}\pi(u-2v)\}.\end{aligned}$$

This is

$$\begin{aligned}&(2M_0-2\Omega_0)(2M+2\Omega)\\ &-(2M_0+2\Omega_0)(2M-2\Omega),\ =8(M_0\Omega-M\Omega_0)\end{aligned}$$

if for a moment

$$M=Q\cos\tfrac{1}{2}\pi u-2Q^5\cos\tfrac{1}{2}\pi u\cos\pi u+Q^9\cos\tfrac{3}{2}\pi u,\qquad M_0=Q-2Q^5+Q^9,$$

$$\Omega=2QS^4\cos\pi u\cos\pi v+A\cos\tfrac{1}{2}\pi(u+2v)+A'\cos\tfrac{1}{2}\pi(u-2v),\ \Omega_0=2QS^4\quad +A+A',$$

or substituting and reducing, the value of $8(M_0\Omega-M\Omega_0)$ to the proper degree of approximation is found to be

$$\begin{aligned}&=-8Q(2QS^4+A+A')\cos\tfrac{1}{2}\pi u\\ &\qquad +8(Q^2S^4+8QA)\cos\tfrac{1}{2}\pi(u+2v)+8(Q^2S^4+8QA')\cos\tfrac{1}{2}\pi(u-2v),\end{aligned}$$

which in virtue of the relations $QA=\Lambda^2S^2$, $QA'=\Lambda'^2S^2$, $Q^2S^2=\Lambda\Lambda'$, is equal to the foregoing value of $c_3c_6\vartheta_{14}\vartheta_{11}$. I have thought it worth while to give this somewhat elaborate verification.

Résumé of the foregoing results.

120. In what precedes we have all the quadric relations between the 16 double theta-functions: or say we have the linear relations between squares (squared functions) and the linear relations between pairs (products of two functions): the number of the asyzygetic linear relations between squares is obviously $=12$; and that of the asyzygetic linear relations between pairs is $=60$ (since each of the 30 tetrads of pairs gives two asyzygetic relations): there are thus in all $12+60$, $=72$ asyzygetic linear relations. But these constitute only a 13-fold relation between the functions, viz.,

they are such as to give for the ratios of the 16 functions expressions depending upon two arbitrary parameters, x, y. Or taking the 16 functions as the coordinates of a point in 15-dimensional space, these coordinates are connected by a 13-fold relation (expressed by means of the foregoing system of 72 quadric equations), and the locus is thus a 13-fold, or two-dimensional, locus in 15-dimensional space.

Hence, taking any four of the functions, these are connected by a single equation; that is regarding the four functions as the coordinates of a point in ordinary space, the locus of the point is a surface.

In particular the four functions may be any four functions belonging to a hexad: by what precedes there is then a linear relation between the squares of the four functions: or the locus is a quadric surface. Each hexad gives 15 such surfaces, or the number of quadric surfaces is $(16\times 15=)$ 240.

The 16-*nodal quartic surfaces.*

121. If the four functions are those contained in any two pairs out of a tetrad of pairs (see the foregoing "Table of the 120 pairs"), then the locus is a quartic surface, which is, in fact, a KUMMER's 16-nodal quartic surface. For if for a moment $x.y$ and $z.w$ are two pairs out of a tetrad, and $r.s$ be either of the remaining pairs of the tetrad; then we have rs a linear function of xy and zw: squaring, r^2s^2 is a linear function of x^2y^2, $xyzw$, z^2w^2; but we then have r^2 and s^2, each of them a linear function of x^2, y^2, z^2, w^2; or substituting we have an equation of the fourth order, containing terms of the second order in (x^2, y^2, z^2, w^2), and also a term in $xyzw$. It is clear that if instead of $r.s$ we had taken the remaining pair of the tetrad we should have obtained the same quartic equation in (x, y, z, w). And moreover it appears by inspection that if xy and zw are pairs in a tetrad, then xz and yw are pairs in a second tetrad, and xw and yz are pairs in a third tetrad: we obtain in each case the same quartic equation. We have from each tetrad of pairs six sets of four functions (x, y, z, w): and the number of such sets is thus $(\tfrac{1}{3}6.30=)60$: these are shown in the foregoing "Table of the 60 GÖPEL tetrads," viz., taking as coordinates of a point the four functions in any tetrad of this table, the locus is a 16-nodal quartic surface.

122. To exhibit the process I take a tetrad 4, 7, 8, 11 containing two odd functions; and representing these for convenience by x, y, z, w, viz.: writing

$$\vartheta_4,\ \vartheta_7,\ \vartheta_8,\ \vartheta_{11}\ (u)=x,\ y,\ z,\ w$$

we have then X, Y, Z, W linear functions of the four squares, viz., it is easy to obtain

$$\begin{aligned}
\alpha(x^2+z^2)-\delta(y^2+w^2)&=2(\alpha^2-\delta^2)\mathrm{X},\\
\delta(\quad,,\quad)-\alpha(\quad,,\quad)&=2(\quad,,\quad)\mathrm{W},\\
-\beta(x^2-z^2)+\gamma(y^2-w^2)&=2(\beta^2-\gamma^2)\mathrm{Y},\\
-\gamma(\quad,,\quad)+\beta(\quad,,\quad)&=2(\quad,,\quad)\mathrm{Z}.
\end{aligned}$$

Also considering two other functions $\vartheta_0(u)$ and $\vartheta_{12}(u)$, or as for shortness I write them, ϑ_0 and ϑ_{12}, we have

$$\vartheta_0^2=\alpha\mathrm{X}+\beta\mathrm{Y}+\gamma\mathrm{Z}+\delta\mathrm{W},$$
$$\vartheta_{12}^2=\alpha\mathrm{X}-\beta\mathrm{Y}-\gamma\mathrm{Z}+\delta\mathrm{W},$$

and substituting the foregoing values of X, Y, Z, W, we find

$$\mathrm{M}\vartheta_0^2=\mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}z^2+\mathrm{D}w^2,$$
$$\mathrm{M}\vartheta_{12}^2=\mathrm{C}x^2+\mathrm{D}y^2+\mathrm{A}z^2+\mathrm{B}w^2,$$

where writing down the values first in terms of α, β, γ, δ and then in terms of the c's, we have

$$\begin{aligned}
\mathrm{M}&=(\alpha^2-\delta^2)(\beta^2-\gamma^2) &&=\tfrac{1}{4}\ .c_8^4-c_4^4,\\
\mathrm{A}&=\beta^2\delta^2-\alpha^2\gamma^2 &&=\text{,,}\ -c_2^2c_6^2,\\
\mathrm{B}&=-\alpha\delta(\beta^2-\gamma^2)+\beta\gamma(\alpha^2-\delta^2) &&=\text{,,}\ \ c_3^2c_4^2-c_{15}^2c_8^2,\\
\mathrm{C}&=\alpha^2\beta^2-\gamma^2\delta^2 &&=\text{,,}\ \ c_1^2c_9^2,\\
\mathrm{D}&=-\alpha\delta(\beta^2-\gamma^2)-\beta\gamma(\alpha^2-\delta^2) &&=\text{,,}\ \ c_{15}^2c_4^2-c_3^2c_8^2\,;
\end{aligned}$$

and we then have further

$$c_4c_8\vartheta_0\vartheta_{12}=c_0c_{12}\vartheta_4\vartheta_8+c_3c_{15}\vartheta_7\vartheta_{11},$$

that is

$$c_4c_8\vartheta_0\vartheta_{12}=c_0c_{12}xz\ \ +c_3c_{15}yw\,;$$

whence equating the two values of $\vartheta_0^2\vartheta_{12}^2$ we have the required quartic equation in x, y, z, w.

123. But the reduction is effected more simply if instead of the c's we introduce the rectangular coefficients a, b, c, &c. We then have

$$\mathrm{M}=(c''^2-b'^2),\quad \mathrm{A}=-a''c,\quad \mathrm{C}=a'b,$$
$$\mathrm{B}=-b'c'-b''c'',\ =bc\,;\ \mathrm{D}=b'b''+c'c'',\ =a'a'',$$

and the equations become

$$\begin{aligned}
(c''^2-b'^2)\vartheta_0^2&=-a''cx^2\ \ +bcy^2+a'bz^2-a'a''w^2,\\
(c''^2-b'^2)\vartheta_{12}^2&=\ \ a'bx^2-a'a''y^2-a''cz^2+\ \ bcw^2\\
\sqrt{b'c''}\,\vartheta_0\vartheta_{12}&=\ \ \sqrt{a}\ \ xz+\sqrt{-b''c'}\,yw,
\end{aligned}$$

so that the elimination gives

$$\begin{aligned}&b'c''(-a''cx^2+bcy^2+a'bz^2-a'a''w^2)\\ &\times(\quad a'bx^2-a'a''y^2-a''cz^2+bcw^2)=(c''^2-b'^2)^2.\{ax^2z^2-b''c'y^2w^2+2\sqrt{-ab''c'}xyzw\},\end{aligned}$$

viz.: this is

$$\begin{aligned}&-a'a''bb'cc''(x^4+y^4+z^4+w^4)\\ &+a'b'cc''(a''^2+b^2)(x^2y^2+z^2w^2)\\ &+\{b'c''(a'^2b^2+a''^2c^2)-\quad a(b'^2-c''^2)^2\}x^2z^2\\ &+\{b'c''(a'^2a''^2+b^2c^2)+b''c'(b'^2-c''^2)^2\}y^2w^2\\ &-a''bb'c''(a'^2+c^2)(x^2w^2+y^2z^2)\\ &-2(b'^2-c''^2)^2\sqrt{-ab''c'}xyzw=0.\end{aligned}$$

124. In this equation the coefficients of x^2z^2 and y^2w^2 are each $=a'a''bc(b'^2+c''^2)$, as at once appears from the identities

$$\begin{cases}a'b.b'-c''.a''c=a(b'^2-c''^2),\\ a'b.c''-b'.a''c=\quad(b'^2-c'')^2,\end{cases}$$

$$\begin{cases}a'a''.b'-c''.bc=-b''(b'^2-c''^2),\\ a'a''.c-b'.bc=\quad c'\,(b'^2-c''^2),\end{cases}$$

by multiplying together in each pair the left hand and the right hand sides respectively. Substituting and dividing by $-a'a''bb'cc''$, we have

$$\begin{aligned}&x^4+y^4+z^4+w^4\\ &-\frac{a''^2+b^2}{a''b}(x^2y^2+z^2w^2)-\frac{b'^2+c''^2}{b'c''}(x^2z^2+y^2w^2)+\frac{a'^2+c^2}{a'c}(x^2w^2+y^2z^2)\\ &+\frac{2(b'^2-c''^2)^2\sqrt{-ab''c'}}{a'a''bb'cc''}xyzw=0\ ;\end{aligned}$$

or if we herein restore the c's in place of the rectangular coefficients this is

$$\begin{aligned}&x^4+y^4+z^4+w^4\\ &-\frac{c_1{}^4+c_2{}^4}{c_1{}^2c_2{}^2}(x^2y^2+z^2w^2)-\frac{c_4{}^4+c_8{}^4}{c_4{}^2c_8{}^2}(x^2z^2+y^2w^2)+\frac{c_6{}^4+c_9{}^4}{c_6{}^2c_9{}^2}(x^2w^2+y^2z^2)\\ &+2\frac{c_0c_3c_{12}c_{15}(c_4{}^4-c_8{}^4)^2}{c_1{}^2c_2{}^2c_4{}^2c_6{}^2c_8{}^2c_9{}^2}xyzw=0,\end{aligned}$$

which is the equation of the 16-nodal quartic surface.

Substituting for x, y, z, w their values ϑ_4, ϑ_7, ϑ_8, $\vartheta_{11}(u)$, we have the equation connecting the four theta-functions 4, 7, 8, 11 of a GÖPEL tetrad. And there is an equation of the like form between the four functions of any other GÖPEL tetrad: for obtaining the actual equations some further investigation would be necessary.

The xy-expressions of the theta-functions.

125. The various quadric relations between the theta-functions, admitting that they constitute a 13-fold relation, show that the theta-functions may be expressed as proportional to functions of two arbitrary parameters x, y; and two of these functions being assumed at pleasure the others of them would be determinate; we have of course (though it would not be easy to arrive at it in this manner) such a system in the foregoing expressions of the 16 functions in terms of x, y; and conversely these expressions must satisfy identically the quadric relations between the theta-functions.

126. To show that this is so as to the general form of the equations, consider first the xy-factors $\sqrt{a}$, $\sqrt{ab}$, &c. As regards the squared functions $(\sqrt{ab})^2$, we have for instance

$$(\sqrt{ab})^2=\frac{1}{\theta^2}\{\mathrm{abfc}_{,}\mathrm{d}_{,}\mathrm{e}_{,}+\mathrm{a}_{,}\mathrm{b}_{,}\mathrm{f}_{,}\mathrm{cde}+2\sqrt{\mathrm{XY}}\},$$

$$(\sqrt{cd})^2=\frac{1}{\theta^2}\{\mathrm{cdfa}_{,}\mathrm{b}_{,}\mathrm{e}_{,}+\mathrm{c}_{,}\mathrm{d}_{,}\mathrm{f}_{,}\mathrm{abe}+2\sqrt{\mathrm{XY}}\},$$

each of these contains the same irrational part $\frac{2}{\theta^2}\sqrt{\mathrm{XY}}$, and the difference is therefore rational; and it is moreover integral, for we have

$$(\sqrt{ab})^2-(\sqrt{cd})^2=\frac{1}{\theta^2}(\mathrm{abc}_{,}\mathrm{d}_{,}-\mathrm{a}_{,}\mathrm{b}_{,}\mathrm{cd})(\mathrm{fe}_{,}-\mathrm{f}_{,}\mathrm{e}),$$

where each factor divides by θ, and consequently the product by θ^2; the value is in fact

$$=(e-f)\begin{vmatrix} 1, & x+y, & xy \\ 1, & a+b, & ab \\ 1, & c+d, & cd \end{vmatrix},$$

a linear function of 1, $x+y$, xy; and this is the case as regards the difference of any two of the squares $(\sqrt{ab})^2$, $(\sqrt{ac})^2$, &c.; hence selecting any one of these squares for instance $(\sqrt{de})^2$, any other of the squares is of the form

$$\lambda+\mu(x+y)+\nu xy+\rho(\sqrt{de})^2;\ (\rho=1)$$

and obviously, the other squares $(\sqrt{a})^2$, &c., are of the like form, the last coefficient ρ

being $=0$. We hence have the theorem that each square can be expressed as a linear function of any four (properly selected) squares.

127. But we have also the theorem of the 16 Kummer hexads.

Obviously the six squares

$$(\sqrt{a})^2,\ (\sqrt{b})^2,\ (\sqrt{c})^2,\ (\sqrt{d})^2,\ (\sqrt{e})^2,\ (\sqrt{f})^2$$

are a hexad, viz.: each of these is a linear function of 1, $x+y$, xy, and therefore selecting any three of them, each of the remaining three can be expressed as a linear function of these.

But further the squares $(\sqrt{a})^2$, $(\sqrt{b})^2$, $(\sqrt{ab})^2$, $(\sqrt{cd})^2$, $(\sqrt{ce})^2$, $(\sqrt{de})^2$ form a hexad. For reverting to the expression obtained for $(\sqrt{ab})^2-(\sqrt{cd})^2$, the determinant contained therein is a linear function of $\mathrm{aa_{,}}$ and $\mathrm{bb_{,}}$, that is of $(\sqrt{a})^2$ and $(\sqrt{b})^2$; we in fact have

$$(a-b).\begin{vmatrix} 1, & x+y, & xy \\ 1, & a+b, & ab \\ 1, & c+d, & cd \end{vmatrix}=(b-c)(b-d)(a-x)(a-y)-(a-c)(a-d)(b-x)(b-y).$$

Hence $(\sqrt{ab})^2-(\sqrt{cd})^2$ is a linear function of $(\sqrt{a})^2$, $(\sqrt{b})^2$; and by a mere interchange of letters $(\sqrt{ab})^2-(\sqrt{ce})^2$, $(\sqrt{ab})^2-(\sqrt{de})^2$, are each of them also a linear function of $(\sqrt{a})^2$ and $(\sqrt{b})^2$; whence the theorem. And we have thus all the remaining 15 hexads.

128. We have a like theory as regards the products of pairs of functions; a tetrad of pairs is of one of the two forms

$$\sqrt{a}\sqrt{b},\ \sqrt{ac}\sqrt{bc},\ \sqrt{ad}\sqrt{bd},\ \sqrt{ae}\sqrt{be}\ \text{and}\ \sqrt{f}\sqrt{ab},\ \sqrt{c}\sqrt{de},\ \sqrt{d}\sqrt{ce},\ \sqrt{e}\sqrt{cd}\,;$$

in the first case the terms are

$$\sqrt{\mathrm{aa_{,}bb_{,}}}\},$$

$$\frac{1}{\theta^2}\{(\mathrm{ab_{,}+a_{,}b})\sqrt{\mathrm{cdefc_{,}d_{,}e_{,}f_{,}}}+(\mathrm{cfd_{,}e_{,}+c_{,}f_{,}de})\sqrt{\mathrm{aa_{,}bb_{,}}}\},$$

$$\frac{1}{\theta^2}\{\quad ,,\qquad\qquad ,,\qquad +(\mathrm{dfc_{,}e_{,}+d_{,}f_{,}ce})\quad ,,\quad\},$$

$$\frac{1}{\theta^2}\{\quad ,,\qquad\qquad ,,\qquad +(\mathrm{efc_{,}d_{,}+c_{,}f_{,}ce})\quad ,,\quad\},$$

and as regards the last three terms the difference of any two of them is a mere constant multiple of $\sqrt{\mathrm{aa_{,}bb_{,}}}$; for instance, the second term $-$ the third term is

$=\frac{1}{\theta^2}(\mathrm{cd}_{,}-\mathrm{c}_{,}\mathrm{d})(\mathrm{fe}_{,}-\mathrm{f}_{,}\mathrm{e})\sqrt{\mathrm{aa}_{,}\mathrm{bb}_{,}},=(c-d)(f-e)\sqrt{\mathrm{aa}_{,}\mathrm{bb}_{,}}$; we have thus a tetrad such that selecting any two terms, each of the remaining terms is a linear function of these.

In the second case the terms are

$$\begin{array}{llll}
\frac{1}{\theta}\{\mathrm{f}\sqrt{\mathrm{abc}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}} & + & \mathrm{f}_{,}\sqrt{\mathrm{a}_{,}\mathrm{b}_{,}\mathrm{cdef}} & \}, \\
\frac{1}{\theta}\{\mathrm{c} \quad ,, & + & \mathrm{c}_{,} \quad ,, & \}, \\
\frac{1}{\theta}\{\mathrm{d} \quad ,, & + & \mathrm{d}_{,} \quad ,, & \}, \\
\frac{1}{\theta}\{\mathrm{e} \quad ,, & + & \mathrm{e}_{,} \quad ,, & \},
\end{array}$$

whence clearly the four terms are a tetrad as above. And it may be added that any linear function of the four terms is of the form

$$\frac{1}{\theta}\{(\lambda+\mu x)\sqrt{\mathrm{abc}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}}+(\lambda+\mu y)\sqrt{\mathrm{a}_{,}\mathrm{b}_{,}\mathrm{cdef}}\}.$$

129. Considering next the actual equations between the squared theta-functions, take as a specimen

$$c_6^2 \quad \vartheta_6^2-c_2^2 \quad \vartheta_2^2+c_1^2 \quad \vartheta_1^2-c_9^2 \quad \vartheta_9^2=0,$$

that is

$$c_6^4(\sqrt{ab})^2-c_2^4(\sqrt{cd})^2+c_1^4(\sqrt{ce})^2-c_9^4(\sqrt{de})^2=0,$$

where $c_6, c_2, c_1, c_9=\sqrt[4]{ab}, \sqrt[4]{cd}, \sqrt[4]{ce}, \sqrt[4]{de}$ respectively. Since the functions $(\sqrt{ab})^2$, &c., contain the same irrational term $\frac{2}{\theta^2}\sqrt{\mathrm{XY}}$, it is clear that the equation can only be true if

$$c_6^4-c_2^4+c_1^4-c_9^4=0,$$

and this being so it will be true if

$$c_2^4\{(\sqrt{ab})^2-(\sqrt{cd})^2\}-c_1^4\{(\sqrt{ab})^2-(\sqrt{ce})^2\}+c_9^4\{(\sqrt{ab})^2-(\sqrt{de})^2\}=0,$$

where by what precedes each of the terms in $\{\}$ is a linear function of $(\sqrt{a})^2$ and $(\sqrt{b})^2$: attending first to the term in $(\sqrt{a})^2$, the coefficient hereof is

$$ef.bc.bd.c_2^4-df.bc.be.c_1^4+cf.bd.be.c_9^4,$$

where for shortness bc, bd, &c., are written to denote the differences $b-c$, $b-d$, &c.: substituting for c_2^4 its value $(\sqrt{cd})^4$, $=cd.cf.df.ab\ ae.be$, and similarly for c_1^4 and c_9^4 their values, $=ce.cf.ef.ab.ad.bd$, and $de.df.ef.ab.ac.bc$ respectively, the whole expression contains the factor $ab.bc.bd.be.cf.df.ef$, and throwing this out, the equation to be verified becomes

$$cd.ae - ce.ad + de.ac = 0$$

which is true identically. The verification is thus made to depend upon that of $c_6{}^4 - c_2{}^4 + c_1{}^4 - c_9{}^4 = 0$; and similarly for the other relations between the squared functions, the verification depends upon relations containing the fourth powers, or the products of squares, of the constants c and k.

130. Among these are included the before-mentioned system of equations involving the fourth powers or the products of squares of only the constants c; and it is interesting to show how these are satisfied identically by the values $c_0 = \sqrt[4]{bd}$, &c.

Thus one of these equations is $c_{12}{}^4 + c_1{}^4 + c_6{}^4 = c_0{}^4$; substituting the values, there is a factor ce which divides out, and the resulting equation is

$$ad.af.df.bc.be + cf.ef.ab.ad.bd + ab.af.bf.cd.de - ac.ae.bd.bf.df = 0.$$

There are here terms in a^2, a, a^0 which should separately vanish; for the terms in a^2 the equation becomes

$$df.bc.be + bd.cf.ef + bf.cd.de - bd.bf.df = 0,$$

which is easily verified; and the equations in a and a^0 may also be verified.

An equation involving products of the squares is $c_{12}{}^2 c_9{}^2 - c_1{}^2 c_4{}^2 + c_3{}^2 c_6{}^2 = 0$. The term $c_{12}{}^2 c_9{}^2$ is here $\sqrt{adf.bce}\ \sqrt{def.abc}$ which is $= \sqrt{(bc)^2 (df)^2 .ab.ac.ad.af.be.ce.de.ef}$, which is taken $= bc.df\sqrt{ab.ac.ad.af.be.ce.de.ef}$; similarly the values of $c_1{}^2 c_4{}^2$ and $c_3{}^2 c_6{}^2$ are $= bd.cf$ and $bf.cd$ each into the same radical, and the equation to be verified is

$$bc.df - bd.cf + bf.cd = 0,$$

which is right: and the other equations may be verified in a similar manner.

131. Coming next to the equations connecting the pairs of theta-functions, for instance

$$c_3 c_{15} \vartheta_0 \vartheta_{12} - c_0 c_{12} \vartheta_3 \vartheta_{15} + c_4 c_8 \vartheta_7 \vartheta_{11} = 0,$$

this is

$$c_3 c_{15} c_0 c_{12} \{ \sqrt{bd}\ \sqrt{ad} - \sqrt{be} \sqrt{ae} \} + c_4 c_8 k_7 k_{11} . \sqrt{b} \sqrt{a} = 0,$$

the products $\sqrt{bd}\,\sqrt{ad}$ and $\sqrt{be}\ \sqrt{ae}$ contain besides a common term the terms $\frac{1}{\theta^2}(\mathrm{dfc_{,}e_{,} + d_{,}f_{,}ce})\sqrt{\mathrm{aa_{,}bb_{,}}}$, and $\frac{1}{\theta^2}(\mathrm{efc_{,}d_{,} + e_{,}f_{,}cd})\sqrt{\mathrm{aa_{,}bb_{,}}}$, hence their difference contains $\frac{1}{\theta^2}(\mathrm{de_{,} - d_{,}e})(\mathrm{fc_{,} - f_{,}c})\sqrt{\mathrm{aa_{,}bb_{,}}}$ which is $= de.fc\sqrt{\mathrm{aa_{,}bb_{,}}}$, that is $de.fc\sqrt{a}\sqrt{b}$: hence the equation to be verified is

$$de.fc.c_3 c_{15} c_0 c_{12} + c_4 c_8 k_7 k_{11} = 0\ ;$$

$c_3 c_{15} c_0 c_{12}$ is $= \sqrt[4]{bef.acd}\ \sqrt[4]{aef.bcd}\ \sqrt[4]{bdf.ace}\ \sqrt[4]{adf.bce}$, where under the fourth root we have 24 factors, which are, in fact, 12 factors twice repeated; and if we write

$\Pi,=ab.ac.ad.ae.af.bc.bd.be.bf.cd.ce.cf.de.df.ef,$ for the product of all the 15 factors, then the 12 factors are in fact all those of Π, except ab, cf, de; viz., we have

$$c_3c_{15}c_0c_{12}=\sqrt[4]{\Pi^2\div(ab)^2(cf)^2(de)^2}.$$

Again, $c_4c_8k_7k_{11},=\sqrt[4]{acf.bde}\ \sqrt[4]{bcf.ade}\ \sqrt[4]{acdef}\ \sqrt[4]{bcdef},$ is a fourth root of a product of 32 factors, which are in fact 16 factors twice repeated, and in the 16 factors, ab does not occur, cf and de occur each twice, and the other 12 factors each once: we thus have

$$c_4c_8k_7k_{11}=\sqrt[4]{\Pi^2(cf)^2(de)^2\div(ab)^2},$$

and the relation to be verified assumes the form

$$fc.de\sqrt[4]{1\div(cf)^2(de)^2}+\sqrt[4]{(cf)^2(de)^2}=0,$$

which, taking $fc.de=-\sqrt[4]{(cf)^4(de)^4}$, is right. And so for the other equations. It will be observed that in the equation $de.fc.c_3c_{15}c_0c_{12}+c_4c_8k_7k_{11}=0$, and the other equations upon which the verifications depend, there is no ambiguity of sign: the signs of the radicals have to be determined consistently with all the equations which connect the c's and the k's: that this is possible appears evident *à priori*, but the actual verification presents some difficulty. I do not here enter further into the question.

Further results of the product-theorem, the $u\pm u'$ formulæ.

132. Recurring now to the equations in $u+u'$, $u-u'$, by putting therein $u'=0$, we can express X, Y, Z, W in terms of four of the squared functions of u, and by putting $u=0$ we can express X′, Y′, Z′, W′ in terms of four of the squared functions of u'; and, substituting in the original equations, we have the products $\vartheta(\ )u+u'.\vartheta(\ )u-u'$ in terms of the squared functions of u and u'.

Selecting as in a former investigation the functions 4, 7, 8, 11 (which were called x, y, z, w) it is more convenient to use single letters for representing the squared functions, and I write

$$\begin{array}{llll}
\vartheta(u+u')\ .\ \vartheta(u-u)' & \vartheta^2u & \vartheta^2u' & \vartheta^20 \\
4\quad 4 = \mathrm{P}, & 4 = p, & 4 = p', & 4 = p_0\ (=c_4^2), \\
7\quad 7 = \mathrm{Q}, & 7 = q, & 7 = q', & 7 = 0, \\
8\quad 8 = \mathrm{R}, & 8 = r, & 8 = r', & 8 = r_0\ (=c_8^2), \\
11\quad 11 = \mathrm{S}, & 11 = s, & 11 = s', & 11 = 0.
\end{array}$$

Then

$$\begin{array}{rcccc} & \mathrm{X} & \mathrm{Y} & \mathrm{Z} & \mathrm{W} \\ \hline \mathrm{P}= & \mathrm{X}' & -\mathrm{Y}' & +\mathrm{Z}' & -\mathrm{W}', \\ \mathrm{Q}= & \mathrm{W}' & -\mathrm{Z}' & +\mathrm{Y}' & -\mathrm{X}', \\ \mathrm{R}= & \mathrm{X}' & +\mathrm{Y}' & -\mathrm{Z}' & -\mathrm{W}', \\ \mathrm{S}= & \mathrm{W}' & +\mathrm{Z}' & -\mathrm{Y}' & -\mathrm{X}', \end{array} \qquad \begin{array}{rcccc} & \mathrm{X} & \mathrm{Y} & \mathrm{Z} & \mathrm{W} \\ \hline p= & \alpha & -\beta & +\gamma & -\delta, \\ q= & \delta & -\gamma & +\beta & -\alpha, \\ r= & \alpha & +\beta & -\gamma & -\delta, \\ s= & \delta & +\gamma & -\beta & -\alpha, \end{array} \qquad \begin{array}{rcccc} & \mathrm{X}' & \mathrm{Y}' & \mathrm{Z}' & \mathrm{W}' \\ \hline p'= & \alpha & -\beta & +\gamma & -\delta, \\ q'= & \delta & -\gamma & +\beta & -\alpha, \\ r'= & \alpha & +\beta & -\gamma & -\delta, \\ s'= & \delta & +\gamma & -\beta & -\alpha. \end{array}$$

Hence

$$\begin{array}{llll} \alpha(p+r)-\delta(q+s)=2(\alpha^2-\delta^2)\mathrm{X}, & & \alpha(p'+r')-\delta(q'+s')=2(\alpha^2-\delta^2)\mathrm{X}', \\ \delta \quad ,, \quad -\alpha \quad ,, \quad =2 \quad ,, \quad \mathrm{W}, & & \delta \quad ,, \quad -\alpha \quad ,, \quad =2 \quad ,, \quad \mathrm{W}, \\ -\beta(p-r)+\gamma(q-s)=2(\beta^2-\gamma^2)\mathrm{Y}, & & -\beta(p'-r')+\gamma(q'-s')=2(\beta^2-\gamma^2)\mathrm{Y}', \\ -\gamma \quad ,, \quad +\beta \quad ,, \quad =2 \quad ,, \quad \mathrm{Z}, & & -\gamma \quad ,, \quad +\beta \quad ,, \quad =2 \quad ,, \quad \mathrm{Z}'. \end{array}$$

and by means of these values

$$\begin{array}{l} 4(\alpha^2-\delta^2)^2\mathrm{X}'\mathrm{X} =\alpha^2(p+r)(p'+r')+\delta^2(q+s)(q'+s')-\alpha\delta[(p+r)(q'+s')+(p'+r')(q+s)], \\ 4 \quad ,, \quad \mathrm{W}'\mathrm{W}=\delta^2 \quad ,, \quad +\alpha^2 \quad ,, \quad -\alpha\delta[\quad ,, \quad ,, \quad], \\ 4(\beta^2-\gamma^2)^2\mathrm{Y}'\mathrm{Y} =\beta^2(p-r)(p'-r')+\gamma^2(q-s)(q'-s')-\beta\gamma[(p-r)(q'-s')+(p'-r')(q-s)], \\ 4 \quad ,, \quad \mathrm{Z}'\mathrm{Z} =\gamma^2 \quad ,, \quad +\beta^2 \quad ,, \quad -\beta\gamma[\quad ,, \quad ,, \quad]. \end{array}$$

Hence

$$\begin{array}{l} 4(\alpha^2-\delta^2)(\mathrm{X}'\mathrm{X}-\mathrm{W}'\mathrm{W})=(p+r)(p'+r')-(q+s)(q'+s'), \\ 4(\beta^2-\gamma^2)(\mathrm{Y}'\mathrm{Y}-\mathrm{Z}'\ \mathrm{Z})=(p-r)(p'-r')-(q-s)(q'-s'), \end{array}$$

and substituting in the expressions for P and R,

$$\begin{array}{l} 4(\alpha^2-\delta^2)(\beta^2-\gamma^2)\mathrm{P}= \\ \qquad (\beta^2-\gamma^2)[(p+r)(p'+r')-(q+s)(q'+s')]-(\alpha^2-\delta^2)[(p-r)(p'-r')-(q-s)(q'-s')], \\ 4 \qquad ,, \qquad \mathrm{R}= \\ \qquad ,, \quad [\quad ,, \qquad ,, \quad]+ \quad ,, \quad [\quad ,, \qquad ,, \quad]. \end{array}$$

Similarly

$$4(\alpha^2-\delta^2)^2\mathrm{W'X}=$$
$$\alpha\delta[(p+r)(p'+r')+(q+s)(q'+s')]-\alpha^2(p+r)(q'+s')-\delta^2(q+s)(p'+r'),$$
$$4\quad\text{,,}\quad \mathrm{X'W}=$$
$$\text{,,}[\quad\text{,,}\qquad\text{,,}\quad]-\delta^2\quad\text{,,}\quad-\alpha^2\quad\text{,,}\quad,$$
$$4(\beta^2-\gamma^2)^2\mathrm{Z'Y}=$$
$$\beta\gamma[(p-r)(p'-r')+(q-s)(q'-s')]-\beta^2(p-r)(q'-s')-\gamma^2(q-s)(p'-r'),$$
$$4\quad\text{,,}\quad \mathrm{Y'Z}=$$
$$\text{,,}[\quad\text{,,}\qquad\text{,,}\quad]-\gamma^2\quad\text{,,}\quad-\beta^2\quad\text{,,}\quad,$$

whence

$$4(\alpha^2-\delta^2)(\mathrm{W'X}-\mathrm{X'W})=-[(p+r)(q'+s')-(p'+r')(q+s)],$$
$$4(\beta^2-\gamma^2)(\mathrm{Z'Y}-\mathrm{Y'Z})=-[(p-r)(q'-s')-(p'-r')(q-s)],$$

and substituting in the expressions for Q and S

$$4(\alpha^2-\delta^2)(\beta^2-\gamma^2)\mathrm{Q}=$$
$$-(\beta^2-\gamma^2)[(p+r)(q'+s')-(p'+r')(q+s)]+(\alpha^2-\delta^2)[p-r)(q'-s')-(p'-r')(q-s)],$$
$$4\quad\text{,,}\quad \mathrm{R}=$$
$$-\quad\text{,,}\quad[\quad\text{,,}\qquad\text{,,}\quad]-\quad\text{,,}\quad[\quad\text{,,}\qquad\text{,,}\quad].$$

133. Hence collecting and reducing

$$4(\alpha^2-\delta^2)(\beta^2-\gamma^2)\mathrm{P}=$$
$$-(\alpha^2-\beta^2+\gamma^2-\delta^2)(pp'-qq'+rr'-ss')+(\alpha^2+\beta^2-\gamma^2-\delta^2)(pr'+p'r-qs'-q's),$$
$$4\quad\text{,,}\quad \mathrm{R}=$$
$$(\alpha^2+\beta^2-\gamma^2-\delta^2)(\quad\text{,,}\quad)-(\alpha^2-\beta^2+\gamma^2-\delta^2)(\quad\text{,,}\quad),$$
$$4\quad\text{,,}\quad \mathrm{Q}=$$
$$(\alpha^2-\beta^2+\gamma^2-\delta^2)(\quad\text{,,}\quad)-(\alpha^2+\beta^2-\gamma^2-\delta^2)(\quad\text{,,}\quad),$$
$$4\quad\text{,,}\quad \mathrm{S}=$$
$$-(\alpha^2+\beta^2-\gamma^2-\delta^2)(\quad\text{,,}\quad)+(\alpha^2-\beta^2+\gamma^2-\delta^2)(\quad\text{,,}\quad);$$

we have $p_0(=c_4^2)=\alpha^2-\beta^2+\gamma^2-\delta^2$, $r_0(=c_8^2)=\alpha^2+\beta^2-\gamma^2-\delta^2$, and thence

$$r_0^2-p_0^2=4(\alpha^2-\delta^2)(\beta^2-\gamma^2);$$

the equations hence become

$$\begin{array}{lrl}
(r_0^2-p_0^2)\mathrm{P} & = & -p_0(pp'-qq'+rr'-ss')+r_0(pr'+p'r-qs'-q's), \\
\quad ,, & \mathrm{R} = & r_0(\qquad ,, \qquad)-p_0(\qquad ,, \qquad), \\
\quad ,, & \mathrm{Q} = & p_0(pq'-p'q+rs'-r's)-r_0(\qquad ,, \qquad), \\
\quad ,, & \mathrm{S} = & -r_0(\qquad ,, \qquad)+p_0(\qquad ,, \qquad).
\end{array}$$

On writing in the equations $u'=0$, then P, Q, R, S, p', q', r', s' become $=p$, q, r, s, p_0, 0, r_0, 0 ; and the equations are (as they should be) true identically. The equations may be written

$$\begin{array}{llllllll}
(c^4\ \ c^4) & \overset{u+u'}{\vartheta} & . \ \overset{u-u'}{\vartheta} & & c^2(\overset{u}{\vartheta}{}^2.\overset{u}{\vartheta}{}^2 \ \ \overset{u}{\vartheta}{}^2.\overset{u}{\vartheta}{}^2 \ \ \overset{u}{\vartheta}{}^2.\overset{u'}{\vartheta}{}^2 \ \ \overset{u}{\vartheta}{}^2.\overset{u}{\vartheta}{}^2) & & c^2(\overset{u}{\vartheta}{}^2.\overset{u}{\vartheta}{}^2 \ \ \overset{u}{\vartheta}{}^2.\overset{u}{\vartheta}{}^2 \ \ \overset{u}{\vartheta}{}^2.\overset{u'}{\vartheta}{}^2 \ \ \overset{u}{\vartheta}{}^2.\overset{u'}{\vartheta}{}^2) \\
\hline
(8-4) & 4 & 4 & = & -4(4.4-7.7+8.8-11.11) & & +8(4.8+8.4-7.11-11.7), \\
(\ ,,\ ) & 8 & 8 & = & +8(\qquad ,, \qquad) & & -4(\qquad ,, \qquad), \\
(\ ,,\ ) & 7 & 7 & = & +4(4.7-7.4+8.11-11.8) & & -8(4.11-11.4+8.7-7.8), \\
(\ ,,\ ) & 11 & 11 & = & -8(\qquad ,, \qquad) & & +4(\qquad ,, \qquad);
\end{array}$$

and there is of course such a system for each of the 60 GÖPEL tetrads.

Differential relations connecting the theta-functions with the quotient-functions.

134. Imagine p, q, r, s, &c., changed into x^2, y^2, z^2, w^2 ; that is, let x, y, z, w represent the theta-functions 4, 7, 8, 11 of u, v; and similarly x', y', z', w' those of u', v', and x_0, 0, z_0, 0 those of 0, 0. Let u', v' be each of them indefinitely small; and take δ, $=u'\frac{d}{du}+v'\frac{d}{dv}$, as the symbol of total differentiation in regard to u, v, the infinitesimals u' and v' being arbitrary : then we have in general

$$\vartheta(u+u',\ v+v)=\vartheta(u,\ v)+\delta\vartheta(u,\ v)+\tfrac{1}{2}\delta^2\vartheta(u,\ v),$$

and hence

$$\mathrm{P}=(x+\delta x+\tfrac{1}{2}\delta^2x)(x-\delta x+\tfrac{1}{2}\delta^2x),\ =x^2+(x\delta^2x-(\delta x)^2),$$

and similarly for Q, R, S. Moreover, observing that x', z' are even functions, y', z' odd functions of $(u',\ v')$, we have

$$x',\ y',\ z',\ w'=x_0+\tfrac{1}{2}\delta^2x_0,\ \delta y_0,\ z_0+\tfrac{1}{2}\delta^2z_0,\ \delta w_0,$$

where δ^2x_0, δy_0, &c. are what δ^2x, δy, &c., become on writing therein $u=0$, $v=0$; δy_0, δw_0 are of course linear functions, δ^2x_0, δ^2z_0 quadric functions of u' and v'. The values of x'^2, y'^2, z'^2, w'^2 are thus $x_0^2+x_0\delta^2x_0$, $(\delta y_0)^2$, $z_0^2+z_0\delta^2z_0$, $(\delta w_0)^2$; and we have

$$\begin{array}{llllcllrrrr}
 & & & & & & & x_0\partial^2x_0 & (\partial y_0)^2 & z_0\partial^2z_0 & (\partial w_0)^2 \\ \cline{8-11}
x^2x'^2 & -y^2y'^2 & +z^2z'^2 & -w^2w'^2 & = & x^2x_0^2 & +z^2z_0^2 & +x^2 & -y^2 & +z^2 & -w^2, \\
x^2y'^2 & -y^2x'^2 & +z^2w'^2 & -w^2z'^2 & = & -y^2x_0^2 & -w^2z_0^2 & -y^2 & +x^2 & -w^2 & +z^2, \\
x^2z'^2 & -y^2w'^2 & +z^2x'^2 & -w^2y'^2 & = & z^2x_0^2 & +x^2z_0^2 & +z^2 & -w^2 & +x^2 & -y^2, \\
x^2w'^2 & -y^2z'^2 & +z^2y'^2 & -w^2x'^2 & = & -w^2x_0^2 & -y^2z_0^2 & -w^2 & +z^2 & -y^2 & +x^2.
\end{array}$$

135. On substituting these values the constant terms (or terms independent of $u,'\ v'$) disappear of themselves; and the equations (transposing the second and third of them) become

$$\begin{array}{rll}
 & & \qquad x_0\partial^2x_0 \qquad\qquad\qquad (\partial y_0)^2 \qquad\qquad\qquad z_0\partial^2z_0 \qquad\qquad\qquad (\partial w_0)^2 \\ \hline
(z_0^4-x_0^4)\{x\partial^2x-(\partial x)^2\} & = & (-x_0^2x^2+z_0^2z^2)+(x_0^2y^2-z_0^2w^2)+(-x_0^2z^2+z_0^2x^2)+(x_0^2w^2-z_0^2y^2), \\
,, \quad \{y\partial^2y-(\partial y)^2\} & = & -(x_0^2y^2-z_0^2w^2)-(-x_0^2x^2+z_0^2z^2)-(x_0^2w^2-z_0^2y^2)-(-x_0^2z^2+z_0^2x^2), \\
,, \quad \{z\partial^2z-(\partial z)^2\} & = & (-x_0^2z^2+z_0^2x^2)+(x_0^2w^2-z_0^2y^2)+(-x_0^2x^2+z_0^2z^2)+(x_0^2y^2-z_0^2w^2), \\
,, \quad \{w\partial^2w-(\partial w)^2\} & = & -(x_0^2w^2-z_0^2y^2)-(-x_0^2z^2+z_0^2x^2)-(x_0^2y^2-z_0^2w^2)-(-x_0^2x^2+z_0^2z^2),
\end{array}$$

where it will be recollected that x, y, z, w mean ϑ_4, ϑ_7, ϑ_8, $\vartheta_{11}\,(u)$; x_0 is $\vartheta_4\,(0)$ that is c_4, and z_0 is $\vartheta_8\,(0)$ that is c_8. But the formulæ contain also

$$\partial^2x_0=(c_4''',\ c_4^{\text{iv}},\ c_4^{\text{v}}\,)\!\!(\,u',\ v')^2, \quad \partial y_0=(c_7',\ c_7''\,)\!\!(\,u',\ v'),$$
$$\partial^2z_0=(c_8''',\ c_8^{\text{iv}},\ c_8^{\text{v}}\,)\!\!(\,u',\ v')^2, \quad \partial w_0=(c_{11}',\ c_{11}''\,)\!\!(\,u',\ v').$$

The formulæ may be written

$$\begin{array}{rcl|l|l|l}
 & & \overbrace{c^2\,.\,\vartheta^2 \quad c^2\,.\,\vartheta^2}^{c_4\partial^2c_4} & \overbrace{c^2\,.\,\vartheta^2 \quad c^2\,.\,\vartheta^2}^{(\partial c_7)^2} & \overbrace{c^2\,.\,\vartheta^2 \quad c^2\,.\,\vartheta^2}^{c_8\vartheta^2c_8} & \overbrace{c^2\,.\,\vartheta^2 \quad c^2\,.\,\vartheta^2}^{(\partial c_{11})^2} \\
 \{\overset{u}{\vartheta}\,.\,\partial^2\overset{u}{\vartheta}-(\partial\overset{u}{\vartheta})^2\} & & & & & \\
(c_8^4-c_4^4)\{4\quad 4\quad 4\} & = & (-4\quad 4\ +8\quad 8) & +(4\quad 7\ -8\ 11) & +(-4\quad 8\ +8\quad 4) & +(4\ 11\ -8\quad 7), \\
,, \quad \{7\quad 7\quad 7\} & = & -(4\quad 7\ -8\ 11) & -(-4\quad 4\ +8\quad 8) & -(4\ 11\ -8\quad 7) & -(-4\quad 8\ +8\quad 4), \\
,, \quad \{8\quad 8\quad 8\} & = & (-4\quad 8\ +8\quad 4) & +(4\ 11\ -8\quad 7) & +(-4\quad 4\ +8\quad 8) & +(4\quad 7\ -8\ 11), \\
,, \quad \{11\ 11\ 11\} & = & -(4\ 11\ -8\quad 7) & +(-4\quad 8\ +8\quad 4) & -(4\quad 7\ -8\ 11) & -(-4\quad 4\ +8\quad 8),
\end{array}$$

where ∂^2c_4, ∂^2c_8, ∂c_7, ∂c_{11} are written in place of ∂^2x_0, ∂^2z_0, ∂y_0, ∂z_0. There is of course a like system of equations for each of the GÖPEL tetrads.

136. Observe that dividing the first equation by $\vartheta_4^2(u)$, or say by ϑ_4^2, the left hand side is a mere constant multiple of $\partial^2\log\vartheta_4$; and the right hand side depends only on the quotient-functions $\vartheta_7\div\vartheta_4$, $\vartheta_8\div\vartheta_4$, $\vartheta_{11}\div\vartheta_4$; each side is a quadric function of (u', v'), and equating the terms in u'^2, $u'v'$, v'^2 respectively, we have

$$\frac{d^2}{du^2}\log\vartheta_4,\quad \frac{d^2}{du\,dv}\log\vartheta_4,\quad \frac{d^2}{dv^2}\log\vartheta_4$$

each of them expressed as a linear function of the squares of the quotient-functions $\vartheta_7\div\vartheta_4$, $\vartheta_8\div\vartheta_4$, $\vartheta_{11}\div\vartheta_4$. The formula is thus a second-deviative formula serving for the expression of a double theta-function by means of three quotient-functions.

Differential relations of the theta-functions.

137. In "The second set of 16," selecting the eight equations which contain Y_1 and W_1, these are

$$\begin{array}{lrrcrrcl}
 & \overset{u+u'}{\vartheta} & .\ \overset{u-u'}{\vartheta} & & \overset{u+u'}{\vartheta} & .\ \overset{u-u'}{\vartheta} & & \overset{(\text{Suffixes } 1.)}{\mathrm{Y} \qquad \mathrm{W}} \\
\tfrac{1}{2}\{ & 4 & 0 & - & 0 & 4\} & = & \overbrace{\mathrm{Y}' \;+ \mathrm{W}'}, \\
 & 12 & 8 & - & 8 & 12 & = & \mathrm{Y}' \;- \mathrm{W}', \\
 & 6 & 2 & - & 2 & 6 & = & \mathrm{W}' + \mathrm{Y}', \\
 & 14 & 10 & - & 10 & 14 & = & \mathrm{W}' - \mathrm{Y}', \\
\hline
\tfrac{1}{2}\{ & 5 & 1 & + & 1 & 5\} & = & \mathrm{X}' \;+ \mathrm{Z}', \\
 & 13 & 9 & + & 9 & 13 & = & \mathrm{X}' \;- \mathrm{Z}', \\
 & 7 & 3 & + & 3 & 7 & = & \mathrm{Z}' \;+ \mathrm{X}', \\
 & 15 & 11 & + & 11 & 15 & = & \mathrm{Z}' \;- \mathrm{X}', \\
\hline
\end{array}$$

and then, considering any line in the upper half and any two lines in the lower half, we can from the three equations eliminate Y_1 and W_1, thus obtaining an equation such as

$$\begin{vmatrix} \vartheta_4\,\vartheta_0-\vartheta_0\vartheta_4, & \mathrm{Y}', & \mathrm{W}' \\ \vartheta_5\,\vartheta_1+\vartheta_1\vartheta_5, & \mathrm{X}', & \mathrm{Z}' \\ \vartheta_{13}\vartheta_9+\vartheta_9\vartheta_{13}, & \mathrm{X}', & -\mathrm{Z}' \end{vmatrix}=0,$$

viz., this is

$$\begin{aligned} &-2\mathrm{X}'\mathrm{Z}' && (\vartheta_4\,\vartheta_0-\vartheta_0\vartheta_4) \\ &+(\ \ \mathrm{X}'\mathrm{W}'+\mathrm{Y}'\mathrm{Z}')&&(\vartheta_5\,\vartheta_1+\vartheta_1\vartheta_5) \\ &+(-\mathrm{X}'\mathrm{W}'+\mathrm{Y}'\mathrm{Z}')&&(\vartheta_{13}\vartheta_9+\vartheta_9\vartheta_{13})=0, \end{aligned}$$

where the arguments of the theta-functions are as above, $u+u'$, $u-u'$, $u+u'$, $u-u'$; and the suffixes of the X', Y', Z', W' are all $=1$.

138. Suppose in this equation u' becomes indefinitely small; if u' were $=0$ the values of X', Y', Z', W' would be α, 0, γ, 0: and hence u' being indefinitely small we take them to be α, $\eth\beta$, γ, $\eth\delta$, where

$$\eth\beta, =\left(u'\frac{d}{du}+v'\frac{d}{dv}\right)\mathrm{Y}, \text{ and } \eth\delta, =\left(u'\frac{d}{du}+v'\frac{d}{dv}\right)\mathrm{W}, \qquad (u=v=0).$$

are in fact linear functions of u' and v'.

We have $\vartheta_4\vartheta_0-\vartheta_0\vartheta_4$ standing for

$$\vartheta_4(u+u')\vartheta_0(u-u')-\vartheta_0(u+u')\vartheta_4(u-u'),$$

and here $\vartheta_4(u\pm u')=\vartheta_4\pm\delta\vartheta_4$, $\vartheta_0(u\pm u')=\vartheta_0\pm\delta\vartheta_0$; the function in question is thus

$$(\vartheta_4+\delta\vartheta_4)(\vartheta_0-\delta\vartheta_0)-(\vartheta_4-\delta\vartheta_4)(\vartheta_0+\delta\vartheta_0)=2\{\vartheta_0\delta\vartheta_4-\vartheta_4\delta\vartheta_0\},$$

where the arguments are (u, v), and the δ denotes $u'\frac{d}{du}+v'\frac{d}{dv}$.

$\vartheta_5\vartheta_1+\vartheta_1\vartheta_5$, that is $\vartheta_5(u+u')\vartheta_1(u-u')+\vartheta_1(u+u')\vartheta_5(u-u')$, becomes simply $=2\vartheta_5\vartheta_1$, and similarly $\vartheta_{13}\vartheta_9+\vartheta_9\vartheta_{13}$ becomes $=2\vartheta_{13}\vartheta_9$; and the equation thus is

$$-2\alpha_1\gamma_1(\vartheta_0\delta\vartheta_4-\vartheta_4\delta\vartheta_0)+(\alpha_1\delta\delta_1+\gamma_1\delta\beta_1)\vartheta_5\vartheta_1+(-\alpha_1\delta\delta_1+\gamma_1\delta\beta_1)\vartheta_{13}\vartheta_9=0,$$

where the proper suffix 1 is restored to the α, $\delta\beta$, γ, and $\delta\delta$.

139. The equation shows that the differential combination $\vartheta_0\delta\vartheta_4-\vartheta_4\delta\vartheta_0$ is a linear function of $\vartheta_5\vartheta_1$ and $\vartheta_{13}\vartheta_9$, the coefficients of these products being of course linear functions of u' and v'; writing the equation

$$\vartheta_0\delta\vartheta_4-\vartheta_4\delta\vartheta_0=\mathrm{A}\vartheta_5\vartheta_1+\mathrm{B}\vartheta_{13}\vartheta_9,$$

we can if we please determine the coefficients in terms of the constants c', c'', c''', c^{iv}, c^{v}; viz., taking u, v indefinitely small, we have

$$\begin{aligned}
\vartheta_0&=c_0, & \delta\vartheta_4 &=u'(c_4'''u+c_4^{\text{iv}}v)+v'(c_4^{\text{iv}}u+c_4^{\text{v}}v),\\
\vartheta_4&=c_4, & \delta\vartheta_0 &=u'(c_0'''u+c_0^{\text{iv}}v)+v'(c_0^{\text{iv}}u+c_0^{\text{v}}v),\\
\vartheta_1&=c_1, & \vartheta_5 &= c_5'u+c_5''v,\\
\vartheta_9&=c_9, & \vartheta_{13} &= c_{13}'u+c_{13}''v,
\end{aligned}$$

or substituting, and equating the coefficients of u and v respectively

$$\begin{aligned}
c_0(c_4'''u'+c_4^{\text{iv}}v')-c_4(c_0'''u'+c_0^{\text{iv}}v')&=\mathrm{A}c_1c_5'+\mathrm{B}c_9c_{13}',\\
c_0(c_4^{\text{iv}}u'+c_4^{\text{v}}v')-c_4(c_0^{\text{iv}}u'+c_0^{\text{v}}v')&=\mathrm{A}c_1c_5''+\mathrm{B}c_9c_{13}'',
\end{aligned}$$

which equations give the values of A, B.

140. Disregarding the values of the coefficients, and attending only to the form of the equation

$$\vartheta_0\delta\vartheta_4-\vartheta_4\delta\vartheta_0=\mathrm{A}\vartheta_5\vartheta_1+\mathrm{B}\vartheta_{13}\vartheta_9,$$

this is one of a system of 120 equations; viz.: referring to the foregoing table of the 120 pairs, it in fact appears that taking any pair such as $\vartheta_0\vartheta_4$ out of the upper compartment or the lower compartment of any column of the table, the corresponding differential combination $\vartheta_0\delta\vartheta_4-\vartheta_4\delta\vartheta_0$ is a linear function of any two of the four pairs in the other compartment of the same column.

Differential relation of u, v and x, y.

141. We have as before, in the two notations, the pairs

A . B	11 . 7
C . DE	5 . 9
D . CE	13 . 1
E . CD	14 . 2
F . AB	10 . 6

and from the expressions given above for the four pairs below the line, it is clear that any linear function of these four pairs may be represented by

$$(a-b)\frac{1}{\theta}\{(\lambda+\mu y)\sqrt{\mathrm{cdefa}_{,}\mathrm{b}_{,}}+(\lambda+\mu x)\sqrt{\mathrm{c}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}\mathrm{ab}}\},$$

where λ, μ are constant coefficients, and the factor $(a-b)$ has been introduced for convenience, as will appear.

We have consequently a relation

$$\sqrt{\mathrm{aa}_{,}}\,\delta\sqrt{\mathrm{bb}_{,}}-\sqrt{\mathrm{bb}_{,}}\,\delta\sqrt{\mathrm{aa}_{,}}=\frac{a-b}{\theta}\{(\lambda+\mu y)\sqrt{\mathrm{cdefa}_{,}\mathrm{b}_{,}}+(\lambda+\mu x)\sqrt{\mathrm{c}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}\mathrm{ab}}\},$$

where as before δ is used to denote $u'\frac{d}{du}+v'\frac{d}{dv}$, u' and v' being arbitrary multipliers; considering u, v as functions of x, y, we have

$$\frac{d}{du}=\frac{dx}{du}\frac{d}{dx}+\frac{dy}{du}\frac{d}{dy},$$
$$\frac{d}{dv}=\frac{dx}{dv}\frac{d}{dx}+\frac{dy}{dv}\frac{d}{dy},$$

and thence $\delta=\mathrm{P}\frac{d}{dx}+\mathrm{Q}\frac{d}{dy}$ if for shortness P, Q are written to denote $u'\frac{dx}{du}+v'\frac{dx}{dv}$ and $u'\frac{dy}{du}+v'\frac{dy}{dv}$ respectively.

142. The left hand side then is

$$=\mathrm{P}\left(\sqrt{\mathrm{aa}_{,}}\frac{d}{dx}\sqrt{\mathrm{bb}_{,}}-\sqrt{\mathrm{bb}_{,}}\frac{d}{dx}\sqrt{\mathrm{aa}_{,}}\right)+\mathrm{Q}\left(\sqrt{\mathrm{aa}_{,}}\frac{d}{dy}\sqrt{\mathrm{bb}_{,}}-\sqrt{\mathrm{bb}_{,}}\frac{d}{dy}\sqrt{\mathrm{aa}_{,}}\right);$$

the coefficients of P and Q are at once found to be

$$=-\tfrac{1}{2}\frac{(\mathrm{a}-\mathrm{b})\sqrt{\mathrm{a}_{,}\mathrm{b}_{,}}}{\sqrt{\mathrm{ab}}},\ -\tfrac{1}{2}\frac{(\mathrm{a}_{,}-\mathrm{b}_{,}\sqrt{\mathrm{ab}}}{\sqrt{\mathrm{a}_{,}\mathrm{b}_{,}}}\text{ respectively,}$$

or observing that $\mathrm{a}-\mathrm{b},=\mathrm{a}_{,}-\mathrm{b}_{,},=a-b$, the equation becomes

$$\mathrm{P}\frac{\sqrt{\mathrm{a}_{,}\mathrm{b}_{,}}}{\sqrt{\mathrm{ab}}}+\mathrm{Q}\frac{\sqrt{\mathrm{ab}}}{\sqrt{\mathrm{a}_{,}\mathrm{b}_{,}}}=-\frac{2}{\theta}\{(\lambda+\mu y)\sqrt{\mathrm{cdefa}_{,}\mathrm{b}_{,}}+(\lambda+\mu x)\sqrt{\mathrm{c}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}\mathrm{ab}}\};$$

or multiplying by $\sqrt{\mathrm{aba}_{,}\mathrm{b}_{,}}$ and writing for shortness $\mathrm{abcdef}=\mathrm{X}$, $\mathrm{a}_{,}\mathrm{b}_{,}\mathrm{c}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}=\mathrm{Y}$, this becomes

$$\mathrm{a}_{,}\mathrm{b}_{,}\{\mathrm{P}+\frac{2}{\theta}(\lambda+\mu y)\sqrt{\mathrm{X}}\}+\mathrm{ab}\{+\mathrm{Q}\frac{2}{\theta}(\lambda+\mu x)\sqrt{\mathrm{Y}}\}=0.$$

143. There are, it is clear, the like equations

$$\mathrm{b}_{,}\mathrm{c}\,\{\mathrm{P}+\frac{2}{\theta}(\lambda'+\mu'y)\sqrt{\mathrm{X}}\ \}+\mathrm{bc}\{\mathrm{Q}+\frac{2}{\theta}(\lambda'+\mu'x)\sqrt{\mathrm{Y}}\ \}=0,$$

$$\mathrm{c}_{,}\mathrm{a}_{,}\{\mathrm{P}+\frac{2}{\theta}(\lambda''+\mu''y)\sqrt{\mathrm{X}}\}+\mathrm{ca}\{\mathrm{Q}+\frac{2}{\theta}(\lambda''+\mu''x)\sqrt{\mathrm{Y}}\}=0,$$

and it is to be shown that $\lambda=\lambda'=\lambda''$ and $\mu=\mu'=\mu''$. For this purpose recurring to the forms

$$\sqrt{\mathrm{aa}_{,}}\delta\sqrt{\mathrm{bb}_{,}}-\sqrt{\mathrm{bb}_{,}}\delta\sqrt{\mathrm{aa}_{,}}=\frac{a-b}{\theta}\{(\lambda+\mu y)\sqrt{\mathrm{cdefa}_{,}\mathrm{b}_{,}}\ \ +(\lambda\ +\mu x\ )\sqrt{\mathrm{c}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}\mathrm{ab}}\},$$

$$\sqrt{\mathrm{bb}_{,}}\delta\sqrt{\mathrm{cc}_{,}}-\sqrt{\mathrm{cc}_{,}}\delta\sqrt{\mathrm{bb}_{,}}=\frac{b-c}{\theta}\{(\lambda'+\mu'y)\sqrt{\mathrm{adefb}_{,}\mathrm{c}_{,}}\ +(\lambda'\ +\mu'x\ )\sqrt{\mathrm{a}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}\mathrm{bc}}\},$$

$$\sqrt{\mathrm{cc}_{,}}\delta\sqrt{\mathrm{aa}_{,}}-\sqrt{\mathrm{aa}_{,}}\delta\sqrt{\mathrm{cc}_{,}}=\frac{c-a}{\theta}\{(\lambda''+\mu''y)\sqrt{\mathrm{bdefc}_{,}\mathrm{a}_{,}}+(\lambda''+\mu''x)\sqrt{\mathrm{b}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}\mathrm{ca}}\},$$

multiply the first equation by $\sqrt{\mathrm{cc}_{,}}$, the second by $\sqrt{\mathrm{aa}_{,}}$, and the third by $\sqrt{\mathrm{bb}_{,}}$, and add: the left hand side vanishes, and therefore the right hand side must also vanish identically.

144. But on the right hand side we have the term $\frac{1}{\theta}\sqrt{\mathrm{defa}_{,}\mathrm{b}_{,}\mathrm{c}_{,}}$ multiplied into

$$(a-b)\mathrm{c}(\lambda+\mu y)+(b-c)\mathrm{a}(\lambda'+\mu' y)+(c-a)\mathrm{b}(\lambda''+\mu'' y),$$

and the term $-\frac{1}{\theta}\sqrt{\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}\mathrm{abc}}$ multiplied into

$$(a-b)\mathrm{c}_{,}(\lambda+\mu x)+(b-c)\mathrm{a}_{,}(\lambda'+\mu' x)+(c-a)\mathrm{b}_{,}(\lambda''+\mu'' y),$$

and it is clear that the whole can only vanish if these two coefficients separately vanish. This will be the case if we have for λ, λ', λ'' the equations

$$\begin{array}{llll}(a-b)\lambda & +(b-c)\lambda' & +(c-a)\lambda'' & =0, \\ c \quad ,, & +a \quad ,, & +b \quad ,, & =0,\end{array}$$

and the like equations for μ, μ', μ''. The equations written down give

$$(a-b)\lambda : (b-c)\lambda' : (c-a)\lambda''=a-b : b-c : c-a$$

that is $\lambda=\lambda'=\lambda''$: and similarly $\mu=\mu'=\mu''$.

145. But this being so, the three equations in P, Q give

$$\mathrm{P}+\frac{2}{\theta}(\lambda+\mu y)\sqrt{\mathrm{X}}=0,\ \mathrm{Q}+\frac{2}{\theta}(\lambda+\mu x)\sqrt{\mathrm{Y}}=0,$$

that is

$$u'\frac{dx}{du}+v'\frac{dx}{dv}=-\frac{2}{x-y}(\lambda+\mu y)\sqrt{\mathrm{X}},$$
$$u'\frac{dy}{du}+v'\frac{dy}{dv}=-\frac{2}{x-y}(\lambda+\mu x)\sqrt{\mathrm{Y}}.$$

In these equations u' and v' are arbitrary; hence λ and μ must be linear functions of u' and v'; say their values are $= \omega u'+\rho v'$, $\sigma u'+\tau v'$ respectively. We have therefore

$$\frac{dx}{du}=-\frac{2}{\theta}(\varpi+\sigma y)\sqrt{\mathrm{X}}\ ,\qquad \frac{dx}{dv}=-\frac{2}{\theta}(\rho+\tau y)\sqrt{\mathrm{X}},$$
$$\frac{dy}{du}=-\frac{2}{\theta}(\varpi+\sigma y)\sqrt{\mathrm{Y}}\ ,\qquad \frac{dy}{dv}=-\frac{2}{\theta}(\rho+\tau x)\sqrt{\mathrm{Y}},$$

or, what is the same thing,

$$-\tfrac{1}{2}\theta\frac{dx}{\sqrt{\mathrm{X}}}=(\varpi+\sigma y)du+(\rho+\tau y)dv,$$
$$-\tfrac{1}{2}\theta\frac{dy}{\sqrt{\mathrm{Y}}}=(\varpi+\sigma x)du+(\rho+\tau x)dv,$$

whence also

$$\sigma du+\tau dv=\ \tfrac{1}{2}\left(\frac{dx}{\sqrt{\mathrm{X}}}-\frac{dy}{\sqrt{\mathrm{Y}}}\right),$$

$$\omega du+\rho dv=-\tfrac{1}{2}\left(\frac{xdx}{\sqrt{\mathrm{X}}}-\frac{ydy}{\sqrt{\mathrm{Y}}}\right),$$

which are the required relations, depending on the square roots of the sextic functions $\mathrm{X}=\mathrm{abcdef}$, and $\mathrm{Y}=\mathrm{a_{,}b_{,}c_{,}d_{,}e_{,}f_{,}}$ of x and y respectively; but containing the constants ϖ, ρ, σ, τ, the values of which are not as yet ascertained.

146. I commence the integration of these equations on the assumption that the values $u=0$, $v=0$ correspond to indefinitely large values of x and y. We have

$$\mathrm{X}=x^6\left(1-\frac{\mathrm{S}}{x}+\ .\ .\ \right),\ \mathrm{Y}=y^6\left(1-\frac{\mathrm{S}}{y}+\ .\ .\ \right),$$

where $\mathrm{S}=a+b+c+d+e+f$; and thence the equations are

$$\sigma du+\tau dv=\ \tfrac{1}{2}\frac{dx}{x^3}\left(1+\frac{\frac{1}{2}\mathrm{S}}{x}\ .\ .\ \right)-\tfrac{1}{2}\frac{dy}{y^3}\left(1+\frac{\frac{1}{2}\mathrm{S}}{y}\ .\ .\ \right),$$

$$\varpi du+\rho dv=-\tfrac{1}{2}\frac{dx}{x^2}\left(1+\frac{\frac{1}{2}\mathrm{S}}{x}\ .\ .\ \right)+\tfrac{1}{2}\frac{dy}{y^2}\left(1+\frac{\frac{1}{2}\mathrm{S}}{y}\ .\ .\ \right),$$

hence integrating

$$\sigma u+\tau v=-\tfrac{1}{2}\left(\frac{1}{x^2}-\frac{1}{y^2}\right)+\ .\ .\ .\ ,$$

$$\varpi u+\rho v=\ \tfrac{1}{2}\left(\frac{1}{x}-\frac{1}{y}\right)+\tfrac{1}{8}\mathrm{S}\left(\frac{1}{x^2}-\frac{1}{y^2}\right),$$

and thence

$$\varpi u+\rho v+\tfrac{1}{4}\mathrm{S}(\sigma u+\tau v)=\tfrac{1}{2}\left(\frac{1}{x}-\frac{1}{y}\right)+\ .\ .\ ,$$

where the omitted terms depend on $\frac{1}{x^3}$, $\frac{1}{y^3}$ &c.

Hence neglecting these terms

$$\frac{\sigma u+\tau v}{\omega u+\rho v+\frac{1}{4}\mathrm{S}(\sigma u+\tau v)}=-\left(\frac{1}{x}+\frac{1}{y}\right),$$

an equation connecting the indefinitely small values of u, v, with the indefinitely large values of x, y.

147. From the equations $\mathrm{A}=k_{11}\omega\sqrt{a}$, $\mathrm{B}=k_7\omega\sqrt{b}$, taking (u, v) indefinitely small and therefore (x, y) indefinitely large, we deduce

$$\frac{c_{11}'u+c_{11}''v}{c_7'u+c_7''v}=\frac{k_{11}}{k_7}\;\frac{1-\frac{1}{2}a\left(\frac{1}{x}+\frac{1}{y}\right)}{1-\frac{1}{2}b\left(\frac{1}{x}+\frac{1}{y}\right)};$$

and hence substituting for $\frac{1}{x}+\frac{1}{y}$ the foregoing value, and introducing an indeterminate multiplier M, we obtain

$$c_{11}'u+c_{11}''v=\mathrm{M}k_{11}\{\varpi u+\rho v+\tfrac{1}{4}\mathrm{S}(\sigma u+\tau v)+\tfrac{1}{2}a(\sigma u+\tau v)\},$$

which breaks up into the two equations

$$c_{11}'=\mathrm{M}k_{11}\{\varpi+(\tfrac{1}{4}\mathrm{S}+\tfrac{1}{2}a)\sigma\},\quad c_{11}''=\mathrm{M}k_{11}\{\rho+(\tfrac{1}{4}\mathrm{S}+\tfrac{1}{2}a)\tau\}$$

and thence also

$$\begin{array}{llllll}
c_7'=\mathrm{M}k_7\{ & ,, & b\ \}, & c_7''=\mathrm{M}k_7\{ & ,, & b\ \},\\
c_5'=\mathrm{M}k_5\{ & ,, & c\ \}, & c_5''=\mathrm{M}k_5\{ & ,, & c\ \},\\
c_{13}'=\mathrm{M}k_{13}\{ & ,, & d\ \}, & c_{13}''=\mathrm{M}k_{13}\{ & ,, & d\ \},\\
c_{14}'=\mathrm{M}k_{14}\{ & ,, & e\ \}, & c_{14}''=\mathrm{M}k_{14}\{ & ,, & e\ \},\\
c_{10}'=\mathrm{M}k_{10}\{ & ,, & f\ \}, & c_{10}''=\mathrm{M}k_{10}\{ & ,, & f\ \},
\end{array}$$

which twelve equations determine the coefficients ϖ, σ, ρ, τ in terms of the c', c'' of the odd functions 5, 7, 10, 11, 13, 14; and moreover give rise to relations connecting these c', c'' with each other and with the constants a, b, c, d, e, f.

148. It is observed that if as before

$$\delta=u'\frac{d}{du}+v'\frac{d}{dv},=\mathrm{P}\frac{d}{dx}+\mathrm{Q}\frac{d}{dy},$$

then, substituting for P, Q their values, we have

$$\begin{aligned}\delta&=-\frac{2}{\theta}(\varpi u'+\rho v')\left(\sqrt{\mathrm{X}}\frac{d}{dx}+\sqrt{\mathrm{Y}}\frac{d}{dy}\right)-\frac{2}{\theta}(\sigma u'+\tau v')\left(y\sqrt{\mathrm{X}}\frac{d}{dx}+x\sqrt{\mathrm{Y}}\frac{d}{dy}\right),\\ &=(\varpi u'+\rho v')\delta_1+(\sigma u'+\tau v')\delta_2,\end{aligned}$$

if for shortness

$$\delta_1=-\frac{2}{\theta}\left(\sqrt{\mathrm{X}}\frac{d}{dx}+\sqrt{\mathrm{Y}}\frac{d}{dy}\right),\ \delta_2=-\frac{2}{\theta}\left(y\sqrt{\mathrm{X}}\frac{d}{dx}+x\sqrt{\mathrm{Y}}\frac{d}{dy}\right),$$

and then operating with $\eth$ on the equations $\mathrm{A}=\omega k_{11}\sqrt{ab}$, &c., we have for instance

$$\mathrm{A}\eth\mathrm{B}-\mathrm{B}\eth\mathrm{A}=\omega^2 k_{11}k_7\{(\varpi u'+\rho v')(\sqrt{a}\eth_1\sqrt{b}-\sqrt{b}\eth_1\sqrt{a})$$
$$+(\sigma u'+\tau v')(\sqrt{a}\eth_2\sqrt{b}-\sqrt{b}\eth_2\sqrt{a})\},$$

which is one of a system of 120 equations, the A, B being in fact any two of the 16 functions.

These are in fact nothing else than the foregoing system of 120 equations giving the values of the differential combinations $\vartheta_0\eth\vartheta_1-\vartheta_1\eth\vartheta_0$, &c., each as a sum of products of pairs of functions, only on the right-hand sides we have expressions such as $\sqrt{a}\eth_1\sqrt{b}-\sqrt{b}\eth_1\sqrt{a}$, &c., which present themselves as perfectly determinate functions of x, y: so that regarding $\varpi u'+\rho v'$, $\sigma u'+\tau v'$ as given linear functions of the arbitrary quantities u', v', there is no longer anything indeterminate in the form of the equations.